Group Theory

Homework
1 Ağustos 1999
Ali Nesin

G denotes a group. For $a, b \in G$ and $X, Y \subseteq G$, we let

$b^a = a^{-1}ba$
$[a, b] = a^{-1}b^{-1}ab$
$[X, Y]$ = Subgroup generated by $\{[x, y]: x \in X, y \in Y\}$
$[a, X] = [\{a\}, X]$
$a^X = \{a^x : x \in X\}$
$C_Y(X) = \{y \in Y : yx = xy \text{ for all } x \in X\}$
$Z(G) = C_G(G)$.

1. Prove or disprove:

1a. $\{f : \mathbb{N} \to \mathbb{N} : f$ is a bijection and $\{x \in \mathbb{N} : f(x) \neq x\}$ is finite$\}$ a subgroup of Sym(**N**).

1b. $\{f : \mathbb{N} \to \mathbb{N} : f$ is a bijection and $|\{x \in \mathbb{N} : f(x) \neq x\}|$ is an even integer$\}$ is a subgroup of Sym(**N**).

1c. $\{q \in \mathbb{Q}^{>0} : q = a/b$ and b is square-free$\}$ a subgroup of $\mathbb{Q}^*$.

2. Find the subgroup of

2a. $\mathbb{Q}^+$ generated by 2/3.

2b. $\mathbb{Q}^+$ generated by 2/3 and 4/9.

2c. $\mathbb{Q}^*$ generated by $\{1/p : p$ a prime in $\mathbb{N}\}$.

2d. $\mathbb{Q}^*$ generated by $\{1/p : p$ an odd prime in $\mathbb{N}\}$.

3. Show that the permutations (1 2) and (1 2 ... n) generate Sym(n).

4. Write the elements of Alt(3) and of Alt(4).

5. Find Z(Sym(4)).

6. Show that $Z(G) \lhd G$.

7. Show that $[X, Y] = [Y, X]$ for all $X, Y \subseteq G$.

8. Let H and K be two normal subgroups of G. Show that $[H, K] \subseteq H \cap K$.

9. Let H be a nonempty finite subset of a group G closed under multiplication. Show that H is a subgroup of G.

10. Let $G/C_G(a)$ denote the right coset space $\{C_G(a)g : g \in G\}$. Show that there is a one-to-one correspondence between $G/C_G(a)$ and a^G.

11. Considering Sym(4) as a subgroup of Sym(5) in a natural way, find $C_{Sym(4)}((25))$.

12. Let $a = (1\ 2)(3\ 4) \in$ Sym(4) $\leq$ Sym(5). Find the set $a^{Sym(4)}$. Find $a^{Sym(5)}$.

13a. Show that for all a, b, c in G,

$$[a, bc] = [a, c]\,[a, b]^c$$
$$[ab, c] = [a, c]^b[b, c].$$

13b. Conclude that if $X \leq G$, then X normalizes the subgroup $[X, Y]$.

13c. Conclude that $[G, X] \lhd G$.

14. Show that if G is finitely generated and $H \lhd G$, then G/H is also finitely generated.

15. For $a \in G$, define the function Inn(a) : $G \to G$ by

$$(\mathrm{Inn}(a))(x) = axa^{-1}.$$

15a. Show that Inn(a) $\in$ Aut(G).

15b. Show that the map Inn : $G \to$ Aut(G) is a homomorphism of groups.

15c. What is the kernel of the homomorphism Inn?

16. Let $H = \{(x, 2x, -x) : x \in \mathbb{R}\}$. Show that $\mathbb{R}^3/H \approx \mathbb{R}^2$.

17. Let G be a group. Let $Z_o(G) = \{1\}$ and

$$Z_{i+1}(G) = \{z \in G : zgz^{-1}g^{-1} \in Z_i \text{ for all } g \in G\}.$$

Show that $(Z_i(G))_i$ is an increasing chain of normal subgroups of G and that

$$Z_{i+1}(G)/Z_i(G) = Z(G/\,Z_i(G)).$$

18. A group G is called **simple** if its only normal subgroups are $\{1\}$ and G. Let $G_1, \ldots, G_n$ be simple groups. Find all the normal subgroups of the Cartesian product $G_1 \times \ldots \times G_n$.

19. Show that $|\,\mathrm{Sym}(n)/\mathrm{Alt}(n)\,| = 2$. (**Hint:** Sym($n$) is generated by $\{(1\ i) : i = 2, 3,\ldots, n\}$).

20. Let $H \lhd G$ and $K \leq G$. Show that $\langle H, K\rangle = HK$.

21. Let $H \lhd G$. Show that $[G/H, G/H] = [G, G]H/H$.

22. Let $G^o = G^{(o)} = G$ and for $i \in \mathbf{N}$, define $G^{i+1} = [G, G^i]$ and $G^{(i+1)} = [G^{(i)}, G^{(i)}]$.

22a. Show that $(G^i)_i$ and $(G^{(i)})_i$ are descending chain of normal subgroups of G.

22b. Show that $G^{(i)} \leq G^i$ for all i.

22c. Show that for $i \leq j$, $(G/G^{(j)})^{(i)} = (G^{(i)}/G^{(j)})$.