MATH 111
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1. Let G and H be two groups and let φ be an isomorphism from G into H. Show that φ^{-1} is an isomorphism from H into G. Conclude that Aut(G) is a group.

2. Let G be a group. Show that $Z(G)$ is a normal subgroup.

3. Let G and H be two groups and let φ be a homomorphism from G into H.

3a. Show that Ker(φ) is a normal subgroup of G.

3b. Show that Im(φ) is a subgroup of G.

3c. Show that Ker(φ) = 1 if and only if φ is one-to-one.

3d. Show that the groups G/Ker(φ) and Im(φ) are isomorhic. (Hint: The map ψ: G/Ker(φ) $\rightarrow$ Im(φ) defined by $\psi(g\text{Ker}(\varphi)) = \varphi(g)$ is a well-defined isomorphism between the two groups).

4. Let G be a group. For $x, y \in G$, define,

4a. For fixed $y \in G$, show that the map $x \rightarrow yxy^{-1}$ is an automorphism of G. Call Inn(y) this automorphism.

4b. Show that the map $y \rightarrow$ Inn(y) is a homomorphism from G into Aut(G). Call Inn this homomorphism.

4c. Show that Ker(Inn) = $Z(G)$.

4d. Show that Inn(G) is a normal subgroup of Aut(G).

5. Let G be a group and $x \in G$. Let $n, m \in \mathbf{Z}$. Set $k = \gcd(n, m)$. Show that the subgroup generated by x^n and x^m is equal to the subgroup generated by x^k.

Definitions:

Let G be a group.

If $X \subseteq G$, the **subgroup generated by** X is the smallest subgroup of G containing X. It exists and it is equal to the intersection of all the subgroups of G that contain X. It is also equal to the set of finite products of elements from X and their inverses.

If H is a subgroup of G, we define the set ***G/H*** as follows:

$$G/H = \{xH : x \in G\}$$

A subgroup H of G is called **normal** if $xH = Hx$ for all $x \in G$, equivalently if $xH \subseteq Hx$ for all $x \in G$. If H is normal in G, then the set G/H becomes a group via the product $(xH)(yH) = xyH$. (If H is not normal, then this product is not a well-defined operation.)

A **homomorphism** from a group G into a group H is a map φ: $G \rightarrow H$ such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$. The kernel **Ker**(φ) of φ is defined to be the set of elements $x \in G$ such that $\varphi(x) = 1$. If φ is one-to-one and onto, then φ is called an **isomorphism**. Then the groups G and H are said to be **isomorphic**. If $G = H$, then φ is called an **automorphism** of G. The set of automorphisms of G is denoted by **Aut**(G).

The **center** of G is the set $\mathbf{Z(G)} := \{z \in G: \forall g \in G\ \ zg = gz\}$.