Math 211 (Algebra)

Work at the Nesin Foundation
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Throughout G stands for a group.

1. Let $H \lhd G$ and $K \leq G$.
1a. Show that $\langle H, K \rangle = HK = KH$.
1b. Show that $H \lhd HK$ and $K \cap H \lhd K$.
1c. Show that $HK/H \approx K/(K \cap H)$.

2. Let H and K be two normal subgroups of G such that $H \cap K = \{1\}$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

3. Let H be another group, $g \in G$ have finite order and $\varphi : G \to H$ a group homomorphism. Show that $o(\varphi(g))$ divides $o(g)$.

4. Show that every ascending chain of subgroups is stationary if and only if every subgroup of G is finitely generated.

5. A subgroup H of G is called **characteristic** if $\varphi(H) = H$ for all $\varphi \in \mathrm{Aut}(G)$.
5a. Show that a characteristic subgroup is a normal subgroup.
5b. Show that a subgroup H of G is characteristic if $\varphi(H) \leq H$ for all $\varphi \in \mathrm{Aut}(G)$.
5c. Show that $Z(G)$ is a characteristic subgroup of G.
5d. Let $X \subseteq G$ be such that $\varphi(X) \subseteq X$ for all $\varphi \in \mathrm{Aut}(G)$. Show that $\langle X \rangle$ is a characteristic subgroup of G.
5e. Show that the subgroup generated by all the minimal subgroups of G is characteristic.

6. For x and y in G, define $x^y = x^{-1}yx$ and $x^G = \{x^y : y \in G\}$. The set x^G is called the **conjugacy class** of x.
6a. Show that the set of conjugacy classes partition G.
6b. What are the elements whose conjugacy class consists of one element?
6c (Reineke). Assume $G = \{1\} \cup x^G$ for $x \in G$. Show that $|G| = 1$ or 2.
6d. What can you say about G if $G = \{1\} \cup x^G \cup y^G$ for some x and $y \in G$?