Field Theory Midterm
2008 November
Ali Nesin

1. Let R be a commutative ring. Let $I \lhd R$. Show that $(R/I)[X] \approx R[X]/I[X]$.
2. Prove Gauss Lemma for $\mathbb{Z}$: Let $f \in \mathbb{Z}[X]$ be irreducible over $\mathbb{Z}$. Show that f is irreducible over $\mathbb{Q}$.
3. Show that $X^3 + 2X^2 + 4X - 6$ is irreducible over $\mathbb{Q}$.
4. Prove Eisenstein Criterion: Let $f(X) = f_0 + f_1X + ... + f_nX^n \in \mathbb{Z}[X]$. Suppose that there is a prime $p \in \mathbb{Z}$ such that
   i) $p \mid a_i$ for $i = 0, 1, ..., n - 1$.
   ii) $p \nmid a_n$.
   iii) $p^2 \nmid a_0$.

   Show that f is irreducible over $\mathbb{Q}$.
5. Show that $X^5 + 2X^3 + (8/7)X^2 - (4/7)X + 2/7$ is irreducible over $\mathbb{Q}$.
6. Concude from above that $2X^5 - 4X^4 + 8X^3 + 14X^2 + 7$ is irreducible over $\mathbb{Q}$.
7. Show that if $p \in \mathbb{N}$ is a prime then $1 + X + X^2 + ... + X^{p-1}$ is irreducible over $\mathbb{Q}$.
8. Let $f \in \mathbb{Z}[X]$ and $p \in \mathbb{Z}$ a prime that does not divide the leading coefficient of f. Show that if the image of f in $\mathbf{F}_p[X]$ is irreducible then so is f.
9. Conclude from above that $7X^4 + 10X^3 - 2X^2 + 4X - 5$ is irreducible over $\mathbb{Z}$.
10. Let K be a field of prime characteristic p. Let $f \in K[X]$. Show that $f(X^p)$ is irreducible over K iff $f(X)$ is irreducible over K and not all the coefficients of f are p-th powers in K.
11. Show that $\mathbb{Q}[\sqrt{2}, \sqrt{5}] = \mathbb{Q}[\sqrt{2} + \sqrt{5}]$. Determine the minimal polynomial of $\sqrt{2} + \sqrt{5}$ over $\mathbb{Q}$, over $\mathbb{Q}[\sqrt{2}]$ and over $\mathbb{Q}[\sqrt{5}]$. Is $\mathbb{Q}[\sqrt{2} + \sqrt{5}]$ Galois over $\mathbb{Q}$?
12. Show that $f(X) = X^3 + X + 1$ is irreducible over $\mathbb{Q}$. Let $\alpha \in \mathbb{C}$ be a root of f. Express α^{-1} and $(\alpha + 2)^{-1}$ as a linear combination of $\mathbb{Q}[\alpha]$.
13. Determine the normal closure of $\mathbb{Q}[2^{1/4}]$ over $\mathbb{Q}$.
14. Show that every field extension of degree 1 and 2 of $\mathbb{Q}$ is a Galois extension. Show that this is not true for extensions of degree > 2.
15. Show that every extension of the form $\mathbb{Q}[\sqrt{a_1}, ..., \sqrt{a_n}]$ where each $a_i \in \mathbb{Q}$ is Galois over $\mathbb{Q}$.
16. Is it true that a Galois extension over a Galois extension of $\mathbb{Q}$ is a Galois extension of $\mathbb{Q}$? Prove or disprove.
17. Show that $\mathbb{Q}[\sqrt{2}, i\sqrt{3}]$ is Galois over $\mathbb{Q}$. Find its Galois group. Find all the subgroups of the Galois group and the corresponding intermediate subfields.