Introduction to Abstract Mathematics
Remake
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Notation: The letters $\boldsymbol{N}$, $\boldsymbol{Z}$, $\boldsymbol{Q}$ and $\boldsymbol{R}$ denote respectively the set of natural numbers, integers, rational numbers and real numbers. Recall that the last three are groups under addition. If X is a set, $P(X)$ denotes the set of subsets of a set.

Note: You should attempt to solve all the questions. An answer with no proof or justification will not be accepted.

Some questions may depend on the previous ones.

1. Is the set of functions

$$\{f: \mathbb{R} \to \mathbb{R} : \lim_{x \to \infty} f(x) \in \mathbb{Q}\}$$

a group under the multiplication of functions?

2. Is the set of functions

$$\{f: \mathbb{R} \to \mathbb{R} : \lim_{x \to \infty} f(x) \in \mathbb{Z}\}$$

a group under the addition of functions?

3. Let G be a finite group and H a subgroup of G. Show that $|H|$ divides $|G|$.

4. Find the subgroup of S_5 generated by (12345) and (12).

5. Let G and H be two groups and f: $G \longrightarrow H$ a homomorphism of groups. Show that if $\text{Im}(f)$ contains a set of generators of H then f is onto.

6. Let G and H be two groups and f: $G \longrightarrow H$ a homomorphism of groups. Define the kernel $\text{Ker}(f)$ of f as follows:

$$\text{Ker}(f) = \{g \in G: f(g) = 1\}.$$

Show that f is one-to-one if and only if $\text{Ker}(f) = \{1\}$.

7. With the above notation show that $\text{Ker}(f)$ is a normal subgroup of G.

8. (16 pts.) Find

$$\lim_{n\to\infty} \sin(n)/n,$$
$$\lim_{n\to\infty} n \sin(1/n),$$
$$\lim_{x\to 0} \sin(2x)/3x$$
$$\lim_{n\to\infty} 3^{n+1}/2^{2n}.$$

9. Find the equation of the line tangent to the graph of the function $y = \cos(3x)$ at the point (0,1).

10. Differantiate the following functions:

$$f(x) = \frac{x^2 - x}{x + 1}$$
$$g(x) = x^2\cos(x)$$

11. Let f: $\mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Find a polynomial $p(x)$ of the form $a + bx + cx^2$ such that $f(0) = p(0)$, $f'(0) = p'(0)$ and $f''(0) = p''(0)$.