Algebra Final Exam
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1. Let p be a prime. Does the subgroup $\oplus_\omega \mathbb{Z}/p\mathbb{Z}$ of $\Pi_\omega \mathbb{Z}/p\mathbb{Z}$ have a complement in $\Pi_\omega \mathbb{Z}/p\mathbb{Z}$? (8 pts.)

2. Let p be a prime. Does the the ideal $\oplus_\omega \mathbb{Z}/p\mathbb{Z}$ of the ring$\Pi_\omega \mathbb{Z}/p\mathbb{Z}$ have a complement (as a ring) in $\Pi_\omega \mathbb{Z}/p\mathbb{Z}$? (9 pts.)

3. Let K be an infinite field. Assume $f \in K[T_1, \ldots, T_n]$ is such that $f(x) = 0$ for all $x \in K^n$. Show that $f = 0$. (10 pts.)

4. Let K be a field. For $A \subseteq K[T_1, \ldots, T_n]$ define,

$$V(A) = \{(x_1, \ldots, x_n) \in K^n : f(x_1, \ldots, x_n) = 0 \text{ for all } f \in A\}.$$

Such a set is called an **algebraic variety** or **an algebraic subset** or **a Zariski closed subset** of K^n.

4a. What are $V(\emptyset)$ and $V(\{1\})$? (2 pts.)

4b. What are the algebraic subsets of K? (4 pts.)

4c. Show that $V(A) = V(\langle A\rangle)$ where $\langle A\rangle$ is the ideal generated by A. (2 pts.)

4d. Show that $V(A) = V(f_1, \ldots, f_k)$ for some $f_1, \ldots, f_k \in K[T_1, \ldots, T_n]$. (4 pts.)

4e. Show that if K real field, then for any $A \subseteq K^n$, $V(A) = V(f)$ for some $f \in K[T_1, \ldots, T_n]$. (3 pts.)

4f. Show that $V(A) \subseteq V(B)$ if $B \subseteq A \subseteq K[T_1, \ldots, T_n]$. (1 pt.)

4g. Show that $\bigcap_i V(A_i) = V\left(\bigcup_i A_i\right)$ for any family A_i of subsets of $K[T_1, \ldots, T_n]$. (1 pt.)

4h. Show that $V(A) \cup V(B) = V(AB)$ for any $A, B \subseteq K[T_1, \ldots, T_n]$. Here AB denotes the set (or the ideal generated by) $\{ab : a \in A, b \in B\}$. (2 pts.)

4i. Call a subset of K^n **Zariski open** if its complement is Zariski closed. Show that the Zariski open subsets of K^n form a topology. (4 pts.)

5. Let R be a commutative ideal with 1. An ideal I of R is called **radical** if for all $r \in R$, $r \in I$ whenever $r^n \in I$ for some natural number n.

5a. Show that the intersection of radical ideals is a radical ideal. Conclude that one can speak of "the radical ideal generated by a subset $A \subseteq R$". (2 pts.)

5b. Let $A \subseteq R$. Show that the radical ideal generated by A is

$$\{r \in R : r^n = \sum_{i=1}^{k} x_i a_i \text{ for some } n, k \in \mathbf{N}, x_1, \ldots, x_k \in R \text{ and } a_1, \ldots, a_n \in A\}.$$

(4 pts.)

5c. Show that I is a radical ideal of R iff R/I has no nilpotent elements. (2 pts.)

6. Let K be a field and $X \subseteq K^n$. Define,

$$I(X) = \{f \in K[T_1, \ldots, T_n] : f(x) = 0 \text{ for all } x \in X\}.$$

6a. Show that $I(X)$ is a radical ideal. (1 pt.)

6b. What is $I(\varnothing)$? (1 pt.)

6c. What can you say about $I(K^n)$? (3 pts.)

6d. What can you say about $I(X)$ if $X \subseteq K$? (4 pts.)

6e. What can you say about $I(X)$ if X has only element? (5 pts.)

6f. Show that $I(X) \subseteq I(Y)$ if $Y \subseteq X \subseteq K^n$. (1 pt.)

6g. Show that $I\left(\bigcup_i X_i\right) = \bigcap_i I(X_i)$ for any family of subsets of K^n. (1 pt.)

6h. Let $X = \{(0, y) : y \in K\}$ and $Y = \{(x, 0) : x \in K\}$. Find $I(X \cup Y)$. (1 pt.)

7. Let K be a field.

7a. Show that $V(A) = V(\sqrt{A})$ for any $A \subseteq K[T_1, \ldots, T_n]$. Here $\sqrt{A}$ denotes the radical ideal generated by A. (2 pts.)

7b. Show that $A \subseteq I(V(A))$ for all $A \subseteq K[T_1, \ldots, T_n]$. Conclude that $\sqrt{A} \subseteq I(V(A))$ for all $A \subseteq K[T_1, \ldots, T_n]$. Give an example of a field K and a radical ideal A where the inclusion is strict. (1+ 2 + 4 pts.)

7c. Show that $X \subseteq V(I(X))$ for all $X \subseteq K^n$. Give an example of a field K and a subset X where the inclusion is strict. (1+4 pts.)

7d. Show that $V(\mathrm{A}) = V(I(V(A)))$ for all $A \subseteq K[T_1, \ldots, T_n]$. (3 pts.)

7e. Show that $I(X) = I(V(I(X)))$ for all $X \subseteq K^n$. (3 pts.)

7f. Let $\mathfrak{R}$ be the set of radical ideals of $K[T_1, \ldots, T_n]$ and Ω the set of Zariski closed subsets of K^n. Show that the map V from $\mathfrak{R}$ into Ω is onto and the map I from Ω into $\mathfrak{R}$ is one-to-one and that $V \circ I = \mathrm{Id}$. (5 pts.)