Set Theory Midterm

Ali Nesin

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I. Let $G = \{f : \mathbb{Z} \rightarrow \mathbb{Z} : \text{for all } x, y \in \mathbb{Z}, f(x + y) = f(x) + f(y)\}$.

1. For $a \in \mathbb{Z}$ define the function $f_a : \mathbb{Z} \rightarrow \mathbb{Z}$ by the formula $f_a(x) = ax$. Show that $f_a \in G$.

2. Show that $f_a + f_b = f_{a+b}$ for any $a, b \in \mathbb{Z}$.

3. Show that $f_a \circ f_b = f_{ab}$ for any $a, b \in \mathbb{Z}$.

4. Let $f \in G$. Show that $f(0) = 0$.

5. Let $f \in G$. Show that for any $n \in \mathbb{Z}$, $f(-n) = -f(n)$.

6. Let $f \in G$. Show that for any $n \in \mathbb{N}$, $f(n) = nf(1)$.

7. Let $f \in G$. Show that if $a = f(1)$, then $f = f_a$.

8. For what integers $a$, is $f_a$ a bijection of $\mathbb{Z}$?

II. Let $G = \{f : \mathbb{Q} \rightarrow \mathbb{Q} : \text{for all } x, y \in \mathbb{Q}, f(x + y) = f(x) + f(y)\}$. As in the previous question, for $a \in \mathbb{Q}$ the function $f_a : \mathbb{Q} \rightarrow \mathbb{Q}$ defined by the formula $f_a(x) = ax$ is in $G$.

1. For what rational numbers $a$, is $f_a$ a bijection of $\mathbb{Q}$?

2. Let $f \in G$. Show that for any $n \in \mathbb{N}$ and $q \in \mathbb{Q}$, $f(nq) = nf(q)$.

3. Show that if $f \in G$ and $a = f(1)$ then $f = f_a$.

III. Let $X$ be a set. We will show that there is no surjection from $X$ into its set of subsets $\mathcal{P}(X)$. In order to get a contradiction, we assume that there is such an $f$. Let $A = \{x \in X : x \notin f(x)\}$.

Let $a \in X$ such that $f(a) = A$. By thinking about the question whether or not $a$ is an element of $A$ arrive at a contradiction.

IV. Let $a, b, c, d$ be four sets. Suppose $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Show that $a = c$ and $b = d$.

V. Let $X$ be a set and $\equiv$ an equivalence relation defined on $X$. Show that the equivalence classes are disjoint.
VI. Let the relation $\equiv$ be defined on $\mathbb{R}$ by

$$x \equiv y \iff x - y \in \mathbb{Z}.$$ 

1. Show that $\equiv$ is an equivalence relation on $\mathbb{R}$.

2. Find a set of representatives of $\mathbb{R}/ \equiv$.

3. Show that if $x \equiv x'$ and $y \equiv y'$ then $x + y \equiv x' + y'$.

4. Is it true that if $x \equiv x'$ and $y \equiv y'$ then $xy \equiv x'y'$ holds for any $x, y \in \mathbb{R}$?