Set Theory Midterm

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I. Let $G = \{f : \mathbb{Z} \longrightarrow \mathbb{Z} : \text{ for all } x, y \in \mathbb{Z},\ f(x+y) = f(x) + f(y)\}$.

1. For $a \in \mathbb{Z}$ define the function $f_a : \mathbb{Z} \longrightarrow \mathbb{Z}$ by the formula $f_a(x) = ax$. Show that $f_a \in G$.
2. Show that $f_a + f_b = f_{a+b}$ for any $a, b \in \mathbb{Z}$.
3. Show that $f_a \circ f_b = f_{ab}$ for any $a, b \in \mathbb{Z}$.
4. Let $f \in G$. Show that $f(0) = 0$.
5. Let $f \in G$. Show that for any $n \in \mathbb{Z}$, $f(-n) = -f(n)$.
6. Let $f \in G$. Show that for any $n \in \mathbb{N}$, $f(n) = nf(1)$.
7. Let $f \in G$. Show that if $a = f(1)$, then $f = f_a$.
8. For what integers a, is f_a a bijection of $\mathbb{Z}$?

II. Let $G = \{f : \mathbb{Q} \longrightarrow \mathbb{Q} : \text{ for all } x, y \in \mathbb{Q},\ f(x+y) = f(x)+f(y)\}$. As in the previous question, for $a \in \mathbb{Q}$ the function $f_a : \mathbb{Q} \longrightarrow \mathbb{Q}$ defined by the formula $f_a(x) = ax$ is in G.

1. For what rational numbers a, is f_a a bijection of $\mathbb{Q}$?
2. Let $f \in G$. Show that for any $n \in \mathbb{N}$ and $q \in \mathbb{Q}$, $f(nq) = nf(q)$.
3. Show that if $f \in G$ and $a = f(1)$ then $f = f_a$.

III. Let X be a set. We will show that there is no surjection from X into its set of subsets $\wp(X)$. In order to get a contradiction, we assume that there is such an f. Let

$$A = \{x \in X : x \notin f(x)\}.$$

Let $a \in X$ such that $f(a) = A$. By thinking about the question whether or not a is an element of A arrive at a contradiction.

IV. Let a, b, c, d be four sets. Suppose $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$. Show that $a = c$ and $b = d$.

V. Let X be a set and $\equiv$ an equivalence relation defined on X. Show that the equivalence classes are disjoint.

VI. Let the relation $\equiv$ be defined on $\mathbb{R}$ by

$$x \equiv y \Leftrightarrow x - y \in \mathbb{Z}.$$

1. Show that $\equiv$ is and equivalence relation on rr.
2. Find a set of representatives of $\mathbb{R}/\equiv$.
3. Show that if $x \equiv x'$ and $y \equiv y'$ then $x + y \equiv x' + y'$.
4. Is it true that if $x \equiv x'$ and $y \equiv y'$ then $xy \equiv x'y'$ holds for any x, $y \in \mathbb{R}$?