Algebra (Group Theory) Midterm

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I. Let p be a prime.

1. Show that for any $k \in \mathbb{N}$, $(1+p)^k \equiv 1 + pk \bmod p^2$. Conclude that $(1+p)^p \equiv 1 \bmod p^2$. Conclude that the formula $\sigma_k(x) = (1+pk)x$ defines an automorphism of $\mathbb{Z}/p^2\mathbb{Z}$ of order p. Show that $\sigma_1^k = \sigma_k$.

2. Show that any automorphism of order p of $\mathbb{Z}/p^2\mathbb{Z}$ is one of the automorphisms σ_k defined in the previous question for some $k = 1, 2, \ldots, p-1$.

II. Let G be a group and $a \in Z_2(G)$ (i.e. $[a, G] \subseteq Z(G)$).

1. Show that for any $x, y \in G$, $[a, xy] = [a, x][a, y]$.

2. Show that for any integer n we have $[a, x^n] = [a, x]^n$.

III. Let G be a group, $a, b \in G$ and $n \in \mathbb{N}$.

1. Show that $(ab)^n = a^n b^n [b, a^{n-1}]^{b^{n-1}} [b, a^{n-2}]^{b^{n-2}} \cdots [b, a]^b$.

2. Conclude that if $[b, a] \in Z(G)$ then $(ab)^n = a^n b^n [b, a^{n-1}][b, a^{n-2}] \cdots [b, a]$.

3. Assume $b \in Z_2(G)$, by using Question I show that
$$(ab)^n = a^n b^n [b, a^{1+2+\cdots+(n-1)}] = a^n b^n [b, a]^{n(n-1)/2}.$$

4. Assume $b \in Z_2(G)$ and that $Z(G)$ is a subgroup of exponent p for a prime $p > 2$. By using the previous question show that $(ab)^p = a^p b^p$.

5. Assume that $G = Z_2(G)$ and that $Z(G)$ is a subgroup of exponent p for a prime $p > 2$. Show that the map defined by $x \mapsto x^p$ is a homomorphism of G.

IV. Let G be a nonabelian group of order p^3 where p is a prime > 2..

1. Show that $Z(G)| \simeq \mathbb{Z}/p\mathbb{Z}$ and $G/Z(G) \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

2. Show that $g^p \in Z(G)$ for any $g \in G$.

3. Show that for any $g \in G$, $\deg g$ is either p or p^2.

4. Show that $G' = Z(G)$.

5. Show that $G = \langle x, y, z \rangle$ and that
$$G = \{x^a y^b z^c : a, b, c \in \{0, 1, \ldots, p-1\}\}.$$

6. Assume x and y both have degree p^2. In this case we may assume without loss of generality that $z = x^p$. Show that there is an $i = 1, 2, \ldots, p-1$ such that $y^p = (x^p)^i = x^{pi}$. By using Question III.5 show that $x^{-i}y$ has order p. By showing that $G/Z(G) = \langle \overline{x}, \overline{x^{-i}y} \rangle$ show that we may assume y has order p. Conclude that $G = \langle x \rangle \rtimes \langle y \rangle \simeq \mathbb{Z}/p^2\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$. Notice at this point that in order to finish the classification in this case we need to find automorphisms of $\mathbb{Z}/p^2\mathbb{Z}$ of order p, which is done in Question I.