# Analysis II, Spring 2013 Midterm

Ali Nesin

April 6, 2013

Part I. Let $X \subseteq \mathbb{R}$. A ***cover*** of $X$ is a family of open intervals $(U_i)_{i \in I}$ such that $X \subseteq \bigcup_{i \in I} U_i$. If $J \subseteq I$ is such that $(U_j)_{j \in J}$ is a cover of $X$ then we say that it is a ***subcover*** of $(U_i)_{i \in I}$. If $I$ is finite we say that $(U_i)_{i \in I}$ is a ***finite cover*** of $X$. A subset $X \subset \mathbb{R}$ is called ***compact*** if any cover of $X$ has a finite subcover.

1. (2 pts.) Show that finite subsets of $\mathbb{R}$ are compact.
2. (3 pts.) Show that the union of any two compact subsets of $\mathbb{R}$ is a compact subset of $\mathbb{R}$.
3. (5 pts.) Show that a compact subset of $\mathbb{R}$ must be bounded.
4. (5 pts.) Show that the intervals (0, 1) and (0, 1] are not compact.
5. (20 pts.) Show that the interval [0, 1] is compact. (Hint: Assume not. By Question 2, either [0, 1/2] or [1/2, 1] is not compact. Choose one of them and keep going.)

Part II. Let $A \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say that $a$ is a ***limit point*** of $A$ if the intersection of $A$ with every open interval containing $a$ contains a point different from $a$.

1. (2 pts.) Show that the set of limit points of (0, 1) is [0, 1].
2. (2 pts.) Find the set of limit points of $\{1/n : n = 1, 2, 3, \ldots\}$.
3. (2 pts.) Find the set of limit points of $\mathbb{Z}$.
4. (2 pts.) Find the set of limit points of $\mathbb{Q}$.
5. (5 pts.) Let $A \subseteq \mathbb{R}$ be such that every rational number is a limit point of $A$. Show that every real number is a limit point of $A$.
6. (10 pts.) Let $A \subseteq \mathbb{R}$. Let $B$ be the set of limit points of $A$. Show that every limit point of $B$ is in $B$.
7. (15 pts.) Show that $a$ is a limit point of $A \subseteq \mathbb{R}$ if and only if there is a sequence whose terms are all distinct that converges to $a$.

8. (10 pts.) Let $A \subseteq \mathbb{R}$ be bounded above. Let $\mathcal{L}(A)$ be the set of limit points of $A$. Assume $\mathcal{L}(A) \neq \emptyset$. Show that $\mathcal{L}(A)$ is bounded above and that $\sup \mathcal{L}(A) \in \mathcal{L}(A)$.