Algebra II, Mitterm

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Let G be a sharply 2-transitive group acting on a set X of size n. Let $x \in X$, $T = G_x$ and $N = (G \setminus \bigcup_{g \in G} T^g) \cup \{1\}$.

1. Show that $|G| = n^2 - n$. (3 pts.)

2. Show that $|T| = n - 1$. (3 pts.)

3. Show that $T^g \cap T = 1$ if $g \notin T$. (3 pts.)

4. Show that $|N| = n$. (3 pts.)

5. Show that $N \setminus \{1\}$ is the set of elements of G that does not fix a point of X. Conclude that N is a normal subset of G. (4 pts.)

6. Show that if $n \in N \setminus \{1\}$, then $C_G(n) \subseteq N$. (4 pts.)

7. Show that $N \setminus \{1\}$ is one conjugacy class. (5 pts.)

8. Find the size of $C_G(n)$ for $n \in N$. (5 pts.)

9. Conclude that N is an abelian group. (5 pts.)

10. Show that N is an elementary abelian group. (7 pts.)

11. Show that $G = N \rtimes T$. (This means that N is a normal subgroup of G, $G = NT$ and $N \cap T = \{1\}$). (2 pts.)

12. Show that G has always an involution. (3 pts.)

13. Show that T has an involution iff n is odd. (3 pts.)

14. Show that N has an involution iff n is even. (3 pts.)

15. Show that T has at most one involution, in which case this involution must be central in T. (Hint: Assume T has two involutions i and j. Let $y \neq x$ and let g carry (y, iy) onto (y, jy). Then ij^g fixes the points y and iy and g fixes x and y). (6 pts.)

16. Show that G is a group of exponent some prime.