Algebra

(Math 211 for CS)
Final Exam
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1. Show that if A is an abelian group then End(A), the ring of homomorphisms from A into A, is a ring under + and o.

2. Is the ring End($\mathbb{Z} \times \mathbb{Z}$) commutative?

3. What is Hom($\mathbb{Z}/8\mathbb{Z}$**,** $\mathbb{Z}/6\mathbb{Z}$)? More generally, what is Hom($\mathbb{Z}/n\mathbb{Z}$**,** $\mathbb{Z}/m\mathbb{Z}$)? How many elements does it have?

4. Find the kernel and the image of the group homomorphism,

$$\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$$

given by,

$$\varphi(x, y) = x + y.$$

5. Let

$$\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$$

given by,

$$\varphi(x, y) = (x + y, x - y).$$

Find φ^n for $n \in \mathbb{N}$. Show that each of the φ^n is an automorphism.

6. Show that $X^3 + 2X + 1$ is an irreducible polynomial of the ring $\mathbf{F}_5[X]$, but not of $\mathbf{F}_{13}[X]$. (Recall that if p is a prime number, $\mathbf{F}_p$ is the field $\mathbb{Z}/p\mathbb{Z}$).

7. Show that $< 5, X + 2 >$ is a maximal ideal of the ring of polynomials $\mathbb{Z}[X]$.

8. For G a group, define,

$$Z(G) = \{z \in G : \text{for all } g \in G, gz = zg\}$$

and

$$Z_2(G) = \{z \in G : \text{for all } g \in G, gzg^{-1}z^{-1} \in Z(G)\}.$$

Show that $Z(G)$ and $Z_2(G)$ are normal subgroups of G.
Show that $Z_2(G)/Z(G) = Z(G/Z(G))$.