Algebra Retake

Fall 2012
Ali Nesin

January 31, 2013

1. Let n and m be two positive integers. Find all endomorphisms from $\mathbb{Z}/n\mathbb{Z}$ into $\mathbb{Z}/m\mathbb{Z}$. For which n and m is there an epimorphism? For which n and m is there an monomorphism?

2. Prove that if R is a Noetherian domain then so is $R[X]$.

3. Show that a finite field field has p^n many elements for some prime p and some natural number $n > 0$.

4. Let K be a field and $f \in K[X] \setminus \{K\}$. Show that K imbeds into the ring $K[X]/\langle f \rangle$ and so $K[X]/\langle f \rangle$ becomes a vector space over K. Show that $\dim_K K[X]/\langle f \rangle = \deg f$.

5. Let $K \leq L \leq M$ be three fields. Of course each one is a vector space over the smaller subfields. Let $(\lambda_i)_{i \in I}$ be a basis of L as a K-vector space. Let $(\mu_j)_{j \in J}$ be a basis of M as an L-vector space. Show that $(\lambda_i \mu_j)_{i \in I,\, j \in J}$ is a basis of M as a K-vector space.

6. Let $K \leq L$ be a field extension. Assume that $\dim_K L$ is a prime. Show that $L \simeq K[X]/\langle f \rangle$ for an irreducible polynomial of degree $\dim_K L$.

7. Find a field with 8 elements.