Throughout $G$ denotes a group.

For $X, Y \subseteq G$, let $[X, Y]$ denote the subgroup generated by all the elements of the form

$$[x, y] := x^{-1}y^{-1}xy$$

for $x \in X$ and $y \in Y$.

1. Show that $[X, Y] = [Y, X]$.
2. Show that if $H$ and $K$ are subgroups of $G$ that normalize each other then $[H, K] \leq H \cap K$.
3. Show that for $x, y, z \in G$, $[x, y] = [x, z][x, y]$ and $[xy, z] = [x, z][y, z]$. Conclude that if $H, K \leq G$, then $H$ and $K$ normalize the subgroup $[H, K]$. Conclude also that if $A \leq G$ is an abelian subgroup and if $g \in N_G(A)$, then the map $\text{ad}(g) : A \to A$ defined by $\text{ad}(g)(a) = [a, g]$ is a group homomorphism whose kernel is $C_A(g)$.

We define $G^n$ and $G^{(n)}$ by induction on $n$:

$$G^0 = G^{(0)} = G, \quad G^{n+1} = [G, G^n], \quad G^{(n+1)} = [G^{(n)}, G^{(n)}].$$

We let $G^i = G^{1^i} = G^{(1)}$.

4. Show that $G^{n+1} \leq G^n$ and that $G^{(n+1)} \leq G^{(n)}$. Show also that $G^n$ and $G^{(n)}$ are characteristic subgroups of $G$.

5. Show that if $H \trianglelefteq G$ and $G/H$ is abelian then $G' \leq H$. Conversely show that if $G' \leq H \leq G$, then $H \trianglelefteq G$ and $G/H$ is abelian.

6. Show that $[G^i, G^j] \leq G^{i+j+1}$ and $G^{(i)} \leq G^j$ for all $i, j$.

We define $Z_n(G)$ by induction on $n$ as follows: $Z_0(G) = 1$ if $n \leq 0$ and for $n \geq 0$, $Z_{n+1}(G)$ is the unique subgroup of $G$ that contains $Z_n(G)$ such that $Z(G/Z_n(G)) = Z_{n+1}(G)/Z_n(G)$.

7. Show that $Z_n(G)$ is a characteristic subgroup of $G$ for all $n$.

8. Show that $[G^i, Z_j] \leq Z_{j-i-1}$ and that $[Z^{i+1}, G^j] = 1$ for all $i, j$.

A group is said to be solvable if $G^{(n)} = 1$ for some $n$. If $G^{(n)} = 1$ but that $G^{(n-1)} \neq 1$, $G$ is said to be solvable of class $n$.

A group is said to be nilpotent if $G^n = 1$ for some $n$. If $G^n = 1$ but that $G^{n-1} \neq 1$, $G$ is said to be nilpotent of class $n$.

9. Show that a nilpotent group is solvable.

10. Let $G$ be nilpotent of class $n$. Show that $G^{n-1} \leq Z_n$. Conclude that $G = Z_n$.

11. Conversely, assume that $G = Z_n$. Show that $G' \leq Z_{n-1}$. Conclude that $G$ is nilpotent of class $n$.

12. Show that $G$ is nilpotent of class $n$ if and only if $Z_n = G$ and $Z_{n-1} \neq G$. 
