Analysis I Final

Fall 2012
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Recall that if $(x_n)_n$ is a sequence of real numbers

$$\limsup x_n = \lim_{n\to\infty} \sup\{x_m : m > n\}.$$

This may be a real number or ∞. From now on we assume that $\infty + r = \infty$ for all $r \in \mathbb{R}$ and that $r\infty = \infty$ for all $r > 0$.

1. (6 pts.) For positive sequences $(x_n)_n$ and $(y_n)_n$, show that

   $$\limsup x_n y_n \le \limsup x_n \limsup y_n.$$

   Does the quality always hold? What can you say about $\limsup(x_n + y_n)$?

2. (10 pts.) Show that $\limsup x_n = \infty$ if and only if there is a subsequence that diverges to ∞.

3. (10 pts.) Let $\limsup x_n = r \in \mathbb{R}$. Show that for any $s > r$, except may be for finitely many n, we have $x_n < s$.

4. (10 pts.) Show that if $\limsup x_n = x \in \mathbb{R}$ then there is a subsequence that converges to x.

5. (10 pts.) Show that if $\limsup |x_n|^{1/n} < 1$ then the series $\sum x_n$ converges absolutely.

6. (5 pts.) Show that if $\limsup |x_n|^{1/n} > 1$ then the series $\sum x_n$ diverges.

7. (2 pts.) Give an example of a convergent series $\sum x_n$ with $\limsup |x_n|^{1/n} = 1$.

8. (2 pts.) Give an example of a divergent series $\sum x_n$ with $\limsup |x_n|^{1/n} = 1$.

9. (10 pts.) Show that if $\lim x_n$ exists then $\limsup x_n = \lim x_n$.

10. (20 pts.) Let $(x_n)_n$ and $(y_n)_n$ be two positive sequences. Show that if $\lim y_n = y \neq 0$ and $\limsup x_n = x$ then $\limsup x_n y_n = xy$. Investigate what happens if $y = 0$.