Algebra I Midterm

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1a. Find groups with increasing orders which contain a unique involution\(^1\). (5 points)

1b. Show that if a group has a unique involution then this involution is central in the group. (2 points)

1c. For what natural numbers \(n\) can a group have a unique element of order \(n\)? (3 points)

2. Let \(G\) be a torsion group without elements of order \(p\) (a prime). Show that \(G\) is a \(p\)-divisible group. (5 points)

3a. Show that a subgroup of index 2 is necessarily normal in \(G\). (5 points)

3b. Show that the same result is false for all \(n \neq 2\). (5 points)

4a. Show that the intersection of two subgroups of finite index has finite index. (5 points)

4b. Show that a subgroup of finite index has finitely many conjugates. (5 points)

4c. Show that a group that has a proper subgroup of finite index has a normal subgroup of finite index. (5 points)

5. Let \(G\) be a group, \(H\) a finite normal subgroup of \(G\). Show that a subgroup of finite index of \(G\) centralizes \(H\). (5 points)

6. Let \(G\) be a group, \(p\) a prime, \(H \leq G\) a finite subgroup such that \(|H|\) is prime to \(p\), and \(g \in G \setminus H\) such that \(g^p \in H\). Show that \(gH\) contains an element of order \(p\). (10 points)

7. Can a group have exactly two involutions? (20 points)

8. Classify all groups \(G\) (up to isomorphisms) with a subgroup \(H \simeq \mathbb{Z}\) of index 2. (25 points)

\(^1\)An involution is an element of order 2