Algebra I Midterm

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1a. Find groups with increasing orders which contain a unique involution[1]. (5 points)

1b. Show that if a group has a unique involution then this involution is central in the group. (2 points)

1c. For what natural numbers n can a group have a unique elements of order n? (3 points)

2. Let G be a torsion group without elements of order p (a prime). Show that G is a p-divisible group. (5 points)

3a. Show that a subgroup of index 2 is necessarily normal in G. (5 points)

3b. Show that the same result is false for all $n \neq 2$. (5 points)

4a. Show that the intersection of two subgroups of finite index has finite index. (5 points)

4b. Show that a subgroup of finite index has finitely many conjugates. (5 points)

4c. Show that a group that has a proper subgroup of finite index has a normal subgroup of finite index. (5 points)

5. Let G be a group, H a finite normal subgroup of G. Show that a subgroup of finite index of G centralizes H. (5 points)

6. Let G be a group, p a prime, $H \leq G$ a finite subgroup such that $|H|$ is prime to p, and $g \in G \setminus H$ such that $g^p \in H$. Show that gH contains an element of order p. (10 points)

7. Can a group have exactly two involutions? (20 points)

8. Classify all groups G (up to isomorphisms) with a subgroup $H \simeq \mathbb{Z}$ of index 2. (25 points)

[1] An involution is an element of order 2