MATH 113
Self Study Material 2
Set Theory

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Number Theory

1) Let a and $b \in \mathbb{Z}$ with one of them is nonzero. We say that d is the greatest common divisior of a and b which is denoted by $gcd(a, b)$ if
i) $d \geq 0$
ii) d divides both a and b
iii) if c divides both a and b then c divides d
a) Show that $gcd(a, b)$ exists.
b) Show that there are integers x and y such that $ax + by = d$.
c) Let $a = 23023$ and $b = 24871$. Find d , x and y as above.
d) Given integers $a_1, ..., a_n$ define gcd of $a_1, ..., a_n$.

2) Let $a \in \mathbb{Z}$ and p be a prime number which does not divide a. Show that $gcd(a, p) = 1$ which means they are relatively prime or coprime.

3) Let p be a prime number. Show that if p divides the product ab then p divides either a or b.

4) We say that $a \in \mathbb{Z}/n\mathbb{Z}$ is invertible if there is a $b \in \mathbb{Z}/n\mathbb{Z}$ such that $ab = 1$.
a) Show that $a \in \mathbb{Z}/n\mathbb{Z}$ is invertible if and only if a and n are relatively prime.
b) Let $n = 35$ find the inverse of 11.
c) Show that n is prime if and only all elements except 0 in $\mathbb{Z}/n\mathbb{Z}$ are invertible.
d) Find the invertible elements of $(\mathbb{Z}/72\mathbb{Z})$. This set of invertible elements is denoted by $(\mathbb{Z}/72\mathbb{Z})^*$.
e) Let p be a prime. Suppose that $xy = 0$ in $(\mathbb{Z}/p\mathbb{Z})$. Show that either $x = 0$

or $y = 0$.
f) $(10\mathbb{Z}+3) \cap (6\mathbb{Z}+1) = n\mathbb{Z}+k$. Find n and k.

5) Using Fermat's Little Theorem, find the remainder when 37^{126} and 29^{29} are divided by 13.

6)For which primes p is p^2+2 also prime?

7)Let p_n denote the n^{th} prime number.Show that $p_{n+1} \leq p_1...p_n + 1$.Deduce that $p_n \leq 2^{2^{n-1}}$.

8)Show that there are infinitely many primes p of the form $6k+5$.
(Hint:Similar proof for there are infinitely many primes of the form $4k+3$)

9)Show that there are infinitely many x and $y \in \mathbb{N}$ such that x^x divides y^y but x does not divide y.

10)Calculate the sums $\displaystyle\sum_{k=0}^{n} \binom{n}{k}(-2)^k$ and $\displaystyle\sum_{k=0}^{n} k\binom{n}{k}$.

Set Theory

1) Let U be any non-empty set. Let $\phi(x)$ and $\psi(x)$ be two properties (of elements of U). Define

$$U_\phi = \{x \in U : \phi(x)\} \quad and \quad U_\psi = \{x \in U : \psi(x)\}$$

Express the following sets in terms of U_ϕ and U_ψ
a) $\{x \in U : \phi(x) \wedge \psi(x)\}$
b) $\{x \in U : \phi(x) \vee \psi(x)\}$

2) Let A and B be two disjoint sets. A set W is said to be a connection of A and B if the following conditions hold:

i) if $Z \in W$ then there are $x \in A$ and $y \in B$ such that $Z = \{x, y\}$.
ii) For each $x \in A$ there is exactly one $y \in B$ such that $\{x, y\} \in W$.
iii) For each $y \in B$ there is exactly one $x \in A$ such that $\{x, y\} \in W$.

Show that for any two disjoint sets A and B the collection $\Sigma(A, B)$ of all connections of A with B is a set.

3) Let A be a non-empty set, let $\equiv \subseteq A \times A$ be relation. Prove that $\equiv$ is

an equicalance relation if and only if there exists a set Q and a surjection $\pi : A \to Q$ so that

$$x \equiv y \Longleftrightarrow \pi(x) = \pi(y).$$

4)Find a bijection between $\mathbb{N}$ and $\mathbb{Q}$.

5)**Definition:** A subgroup of $\mathbb{Z}$ is a subset of $\mathbb{Z}$ which is closed under substraction.
Find all subgroups of $\mathbb{Z}$.

6)Let K be a field. Show that K has only two ideal namely 0 and K itself.

7)Find all functions f from $\mathbb{N}$ to itself which satisfies $f(x+y) = f(x) + f(y)$.

8)**Filters**
Definition: Let X be a set. A filter on X is a set $\mathcal{F}$ of subsets of X that satisfies:

i) If $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.
ii) If A and B are in $\mathcal{F}$ then $A \cap B \in \mathcal{F}$.
iii) $\emptyset \notin \mathcal{F}$ and $X \in \mathcal{F}$.
If A is a non empty fixed subset of X then the set $\mathcal{F}(A)$ of subsets of X that contains A is a filter on X. Such a filter is called a Principal Filter. If X is infinite then the set of cofinite subsets of X is a called a Frechet Filter. A maximal filter is called an Ultrafilter.

Fix a set X
a) Show that a principal filter $\mathcal{F}(A)$ on X is an ultrafilter if and only if A is a singleton.
b) Show that the Frechet filter (on an infinite set) can not be contained in a principal filter.
c) Let $(\mathcal{F}_i)_{i \in I}$ be a family of filters then $\bigcap_{i \in I} \mathcal{F}_i$ is a filter.
d) Let $\mathcal{F}$ be a set of subsets of X so that for any $A_1, ...A_n \in \mathcal{F}$,
$A_1 \cap ... \cap A_n \neq \emptyset$, then there is a filter that contains $\mathcal{F}$. Describe this filter in terms of $\mathcal{F}$.
e) Show that a filter $\mathcal{F}$ is an ultrafilter if and only if for any $A \subseteq X$ either A, or A^c is in $\mathcal{F}$. Conclude that in an ultrafilter $\mathcal{F}$, if $A \cup B \in \mathcal{F}$ then either A or B is in $\mathcal{F}$.

f) Conclude that any ultrafilter on X that contains a finite subset of X is a pricipal filter. Deduce that every non-principla ultrafilter contains the Frechet filter.