Number Theory

1) (Prime Desert) Show that for all \( n \in \mathbb{N} \) there exists \( A, B \in \mathbb{N} \) with \( A - B \geq n \) such that there is no prime between \( A \) and \( B \).

2) There are infinitely many primes of the form \( 4k + 3 \).

3) For a prime \( p \), if \( p = 3k + 1 \) then \( p = 6m + 1 \).

4) Define \( F_n = 2^{2^n} + 1 \). Show that \( F_0.F_1...F_{n-1} + 2 = F_n \). Deduce that there are infinitely many primes.

5) Is \( 2^{13} - 1 \) prime?

Naive Set Theory and Combinatorics

1) Let \( X \) be a set. Show that there is no onto function from \( X \) to \( \mathcal{P}(X) \).

2) Let \( U \) be any non-empty set
   a) Let \( \phi(x) \) and \( \psi(x) \) be two properties (of elements of \( U \)). Assume that \( \phi(x) \Rightarrow \psi(x) \). Look at the sets

   \[ U_\phi = \{ x \in U : \phi(x) \} \quad \text{and} \quad U_\psi = \{ x \in U : \psi(x) \} \]

   What is the relationship between \( U_\phi \) and \( U_\psi \) ?
   b) Take \( U = \mathbb{Z} \) and find concrete examples for \( \phi \), \( \psi \), \( U_\phi \), and \( U_\psi \).
      (Hint: Consider the relation divides in \( \mathbb{Z} \)).

3) Find a bijection between \( \mathbb{N} \times \mathbb{N} \) and \( \mathbb{N} \) explicitly.
4) Find an infinite family $(A_i)_{i \in \mathbb{N}}$ of sets so that any finite intersection of these sets is non-empty but $\bigcap_{i \in \mathbb{N}} A_i = \emptyset$. And another infinite family $(B_i)_{i \in \mathbb{N}}$ of sets so that $\bigcap_{i \in \mathbb{N}} B_i$ has only one element.

5) Let $\mathcal{P}$ be the power set of $I = \{1, 2, 3, 4, 5\}$. Let $S$ be a randomly chosen subset of $\mathcal{P}$. What is the probability that $S$ is the power set of some subset of $I$?

6) Let $|X| = n$ and $|Y| = m$. Find the number of all bijections from $X$ to $X$ and number of all functions from $X$ to $Y$ and number of all one to one functions from $X$ to $Y$ and number of all onto functions from $X$ to $Y$.

**Axiomatic Set Theory**

1) By using $\mathbb{N}$ is well ordered define the successor function $s$ (Assume you do not know $s$ and define it from the well order of $\mathbb{N}$).

2) Let $A$ and $B$ be two sets. Show that the collection of all functions from $A$ to $B$ is a set.

3) Let $X$ be set and let $\equiv$ be an equivalence relation on $X$. Define the equivalence classes of $X$ under $\equiv$ as $\bar{x} = \{y \in X : x \equiv y\}$. Show that the collection of all the equivalence classes of $X$ under $\equiv$ is a set.

4) Show that there is no $y \in \mathbb{N}$ such that $x < y < S(x)$.

5) Show that there are no $z \in \mathbb{Z}$ such that $z + z = (1, 0)$.

6) Show that for any $a, b \in \mathbb{Q}$ there is an element $c \in \mathbb{Q}$ such that $a < c < b$.

7) Show that there is no $x \in \mathbb{Q}$ such that $x \times x = 2$. 