Algebra

First Midterm
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1. Let G be a set together with an associative binary operation $(x, y) \mapsto xy$.

1a. Assume that for all $a, b \in G$ there are unique $x, y \in G$ such that $ax = ya = b$. Show that G is a group under this binary operation.

1b. Assume that for all $a, b \in G$ there is a unique $x \in G$ such that $ax = b$. Is G necessarily a group under this binary operation?

2. Let A and B be two simple groups[1]. Find all normal subgroups of $A \times B$.

3a. Show that there are subgroups $G_0 = 1 \lhd G_1 \lhd G_2 \lhd G_3 = \mathrm{Sym}(4)$ of $\mathrm{Sym}(4)$ such that G_{i+1}/G_i is abelian.

3b. Conclude that the only simple subgroups of $\mathrm{Sym}(4)$ are cyclic of prime order.

3c. Conclude that a simple nonabelian group cannot have a proper subgroup of index ≤ 4.

3d. Conclude that a simple nonabelian group cannot have a proper subgroup with less than 5 conjugates.

3e. Conclude that, if $i \leq 3$, a group of order $2^i \times 3^n$ cannot be simple and nonabelian.

4. Let A and T be two groups. Let $\varphi : T \to \mathrm{Aut}(A)$ be a homomorphism. For t in T, we will denote the automorphism $\varphi(t)$ by φ_t. On the set $G = A \times T$ we define the following multiplication:

$$(a, t)(b, s) = (a\varphi_t(b), ts).$$

4a. Show that this defines a group operation on G with $(1_A, 1_B)$ as the identity element and with $(\varphi_{t^{-1}}(a^{-1}), t^{-1})$ as the inverse of (a, t).

4b. Assume $\varphi_t = \mathrm{Id}_A$ for all $t \in T$. What can you say about G?

4c. Show that $A_1 = A \times 1$ is a normal subgroup of G and is isomorphic to A.

4d. Show that $T_1 = 1 \times T$ is a subgroup of G and is isomorphic to A.

4e. Show that the two subgroups of A_1 and T_1 intersect trivially.

4f. Show that $G = A_1T_1$.

This group is called the **semidirect product** of A and T and is denoted by $A \rtimes_\varphi T$.

5. Let G be a group and assume that G has two subgroups A and T such that

$$A \lhd G,$$
$$A \cap T = 1,$$
$$G = AT.$$

Let $\varphi : T \to \mathrm{Aut}(A)$ be the group homomorphism given by $\varphi_t(a) = tat^{-1}$ (for $a \in A$, $t \in T$). Show that $G \approx A \rtimes_\varphi T$.

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[1] A group G is called simple if 1 and G are the only normal subgroups of G.