1. Let $G$ be a set together with an associative binary operation $(x, y) \mapsto xy$.

1a. Assume that for all $a, b \in G$ there are unique $x, y \in G$ such that $ax = ya = b$. Show that $G$ is a group under this binary operation.

1b. Assume that for all $a, b \in G$ there is a unique $x \in G$ such that $ax = b$. Is $G$ necessarily a group under this binary operation?

2. Let $A$ and $B$ be two simple groups. Find all normal subgroups of $A \times B$.

3a. Show that there are subgroups $G_0 = 1 \triangleleft G_1 \triangleleft G_2 \triangleleft G_3 = \text{Sym}(4)$ of $\text{Sym}(4)$ such that $G_{i+1}/G_i$ is abelian.

3b. Conclude that the only simple subgroups of $\text{Sym}(4)$ are cyclic of prime order.

3c. Conclude that a simple nonabelian group cannot have a proper subgroup of index $\leq 4$.

3d. Conclude that a simple nonabelian group cannot have a proper subgroup with less than 5 conjugates.

3e. Conclude that, if $i \leq 3$, a group of order $2^i \times 3^n$ cannot be simple and nonabelian.

4. Let $A$ and $T$ be two groups. Let $\varphi : T \to \text{Aut}(A)$ be a homomorphism. For $t$ in $T$, we will denote the automorphism $\varphi(t)$ by $\varphi_t$. On the set $G = A \times T$ we define the following multiplication:

$$(a, t)(b, s) = (a\varphi_t(b), ts).$$

4a. Show that this defines a group operation on $G$ with $(1_A, 1_T)$ as the identity element and with $(\varphi_{t^{-1}}(a^{-1}), t^{-1})$ as the inverse of $(a, t)$.

4b. Assume $\varphi_t = \text{Id}_A$ for all $t \in T$. What can you say about $G$?

4c. Show that $A_1 = A \times 1$ is a normal subgroup of $G$ and is isomorphic to $A$.

4d. Show that $T_1 = 1 \times T$ is a subgroup of $G$ and is isomorphic to $A$.

4e. Show that the two subgroups of $A_1$ and $T_1$ intersect trivially.

4f. Show that $G = A_1T_1$.

This group is called the semidirect product of $A$ and $T$ and is denoted by $A \rtimes_{\varphi} T$.

5. Let $G$ be a group and assume that $G$ has two subgroups $A$ and $T$ such that

$A \triangleleft G,$

$A \cap T = 1,$

$G = AT.$

Let $\varphi : T \to \text{Aut}(A)$ be the group homomorphism given by $\varphi_t(a) = tat^{-1}$ (for $a \in A$, $t \in T$). Show that $G \cong A \rtimes_{\varphi} T$.

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1 A group $G$ is called simple if 1 and $G$ are the only normal subgroups of $G$. 