Introduction to Analysis

Selçuk Demir

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1. Let (M, d) be a metric space. For $x, y \in M$ define
$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$
Show that ρ is also a metric on M. Show that these two metrices are equivalent (i.e. they generate the same open sets).

2. Show that a compact space is bounded.

3. Let A be a closed subset of M. For $x \in M$ define
$$f(x) = d(x, A) = \inf\{d(x, y) : y \in A\}.$$
Show that $d(x, y) \geq |f(x) - f(y)|$ for all $x, y \in M$. Show also that $f(x) = 0$ iff $x \in A$.

4. Suppose that every sequence in M has a convergent subsequence. If $\varepsilon > 0$ is given show that any subset A with the property that $d(x, y) \geq \varepsilon$ for all $x, y \in A$ is finite.

5. Suppose that K is a compact subset of M and B is a closed subset of M with $K \cap B = \emptyset$. Define $d(K, M)$ to be $\inf\{d(k, b) : k \in K, b \in B\}$. Show that $d(K, M) > 0$. Show that one can find metric spaces where A, B are disjoint non-empty closed subsets of M with $d(A, B) = 0$.

6. Suppose that $(x_n)_{n \in \mathbb{N}}$ is a sequence in M converging to some $x \in M$. Show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.

7. Show that, if $K \subset \mathbb{R}^n$ is closed and bounded, every sequence in K has a subsequence which converges to some point of K.

8. Let $K \subset \mathbb{R}^n$. Show that K is bounded iff it is totally bounded.

9. Show that in a general metric space a bounded set is not necessarily totally bounded.

10. Let X and Y be metric spaces. Put $Z = X \times Y$ and

$$d((x,y),(z,t)) = d_X(x,z) + d_Y(y,t)$$

for all $(x,y),(z,t) \in Z$. Show that Z is a metric space with this metric. Show that if $U \subset X$ and $V \subset Y$ then $U \times V$ is open in Z iff $U \in OP(X)$ and $V \in OP(Y)$.

11. Let X be metric space and $\Delta := \{(x,x) : x \in X\}$. Show that Δ is closed in $X \times X$.

12. Let E be any set which is not empty. $X := \ell^\infty(E)$ is defined to be the space of all bounded complex functions on E. For $f, g \in X$ we define

$$d(f,g) = \sup_{p \in E} |f(p) - g(p)| .$$

Show that X is a complete metric space.

13. Let X be real or complex vector space. By a norm on X we mean a function $\|\cdot\| : X \longrightarrow \mathbb{R}$ with the following properties: (a) $\|x\| \geq 0$ for all $x \in X$; (b) $\|x\| = 0$ iff $x = 0$; (c) $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$; (d) $\|ax\| = |a| \|x\|$ for all numbers a and $x \in X$. Show that if X is a normed vector space, it becomes in a natural way a meric space by putting $d(x,y) = \|x - y\|$.

14. For $X \in \mathbb{R}^n$ define $\|X\|_1 = \sum_i |x_i|$ and $\|X\|_2 := \sqrt{\sum_i |x_i|^2}$ and $\|X\|_\infty = \sup_i |x_i|$. All these norms are equivalent in the sense that they generate the same open sets.

15. Let V be a vector space, $\|\|_1$ and $\|\|_2$ two norms on V. Show that these two norms are equivalent iff there exist $M, N > 0$ such that

$$M \|x\|_1 \leq \|x\|_2 \leq N \|x\|_2$$

for all $x \in V$.

16. Let X and Y be metric spaces, $f : X \longrightarrow Y$. We say that f preserves convergence if whenever $x_n \longrightarrow x$ in X, $(f(x_n))_n$ also converges and $f(x_n) \longrightarrow f(x)$. If K is a compact subset of X and f preserves convergence, show that $f(K)$ must be compact.

17. Let U, V be two dense open subsets of a metric space X. Is $U \cap V$ dense in X?

18. Let X, Y be metric spaces. Show that $X \times Y$ is compact iff X and Y are compact.

19. Let X_n be a metric space for every $n \in \mathbb{N}$. Let $X := \prod_{i=1}^{\infty} X_n$ be the space of all sequences $x = (x_n)_n$ in $\cup_{n \in \mathbb{N}} X_n$ such that $x_n \in X_n$ for each $n \in \mathbb{N}$. On X define

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{(1 + d_n(x_n, y_n))}$$

for each $x, y \in X$. Show that X is a metric space. Show also that X is compact iff so is each X_n.

20. Show that a subspace of a separable metric space is separable.

21. Let M be a separable metric space. Show that if $(U_\alpha)_{\alpha \in I}$ is a family of pairwise disjoint non-empty open subsets of M, then I should be countable.

22. Show that if S is infinite, the space $\ell^\infty(S)$ is not separable.

23. If M is a connected metric space and if M contains at least two points, show that M is uncountable.

24. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^d$, $X \in \mathbb{R}^d$. Show that $X_n \longrightarrow X$ iff, for each $i \in [1, n]$ we have $X_n(i) \longrightarrow X(i)$.

25. Let M be a metric space, K a non-empty compact subset of M, $x \in M - K$. Show that there exists a $y \in K$ such that $d(x, K) = d(x, y)$.

26. Let M be a metric space, K a subset of M. Define

$$\text{diam}(K) := \sup\{d(x, y) : x, y \in K\}.$$

Show that K is bounded iff $\text{diam}(K)$ is finite.

27. Show that, if K is compact, then $\text{diam}(K)$ is finite and that there are two points $x, y \in K$ such that $d(x, y) = \text{diam}(K)$.

28. Let M be a complete metric space, $(E_n)_{n \in \mathbb{N}}$ a sequence of closed sets with the following conditions:

    (a) $E_{n+1} \subseteq E_n$ for each $n \in \mathbb{N}$,

(b) $\lim_{n \longrightarrow \infty} \text{diam}(E_n) = 0$.

Show that $\cap_{n \in \mathbb{N}} E_n$ consists of exactly one point.

29. If $A, B \subset \mathbb{R}^d$, we define $A + B$ to be the set of all $X + Y$ where $X \in A$ and $Y \in B$. If A and B are closed balls in $\mathbb{R}^d$, show that $A + B$ is also a closed ball.

30. Show that if A and B are both compact, then $A + B$ is also compact.

31. Show that if A is compact, B is closed, then $A + B$ is closed.

32. If A and B are both closed, give an example to show that $A + B$ is not necessarily closed.

33. Let M be a metric space, $A \subset M$ which is not empty. Then, the function $f : M \longrightarrow \mathbb{R}$ given by $f(x) = d(x, A)$ (for all $x \in M$) is continuous.

34. Let M be a metric space, A, B two non-empty disjoint closed subsets. Show that there exists a continuous function $f : M \longrightarrow [0, 1]$ such that $f(x) = 1$ for all $x \in A$ and $f(x) = 0$ for all $x \in B$.

35. Let M be metric space, A, B two non-empty disjoint closed subsets of M. Show that there exist $U, V \in OP(M)$ such that $A \subset U$, $B \subset V$ and $U \cap V = \emptyset$.

36. A metric space M is said to be locally compact is for every $x \in M$ there is a compact set K such that $x \in K^\circ$. For example, $\mathbb{R}^n$ is a locally compact metric space. If M is a closed subset of $\mathbb{R}^n$, show that it is, as a subspace, a locally compact metric space. What can you say if M is open? What if it is the intersection of an open and a closed subset of $\mathbb{R}^n$? (Such subsets are said to be locally closed.)

37. Let M be a locally compact metric space, K a non-empty compact subset, $U \in OP(M)$ with $K \subset U$. Show that there is a $O \in OP(M)$ such that $\overline{O}$ is compact with $K \subset O \subset \overline{O} \subset U$.

38. Let $X \in \mathbb{R}^N$ and $\varepsilon > 0$. Show that $S(X, \varepsilon)$ is path-connected, hence connected.

39. Let M be a compact metric space, N another metric space, $f : M \longrightarrow N$ a continuous bijection. Show that f is a homeomorphism. I.e., f^{-1} is also continuous.

40. Let K be a compact metric space, $f : K \longrightarrow K$ an isometry. This means that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in M$. Show that f is a bijection.

41. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as follows: If $x \in \mathbb{R} - \mathbb{Q}$, then $f(x) = 0$. If $x = \frac{m}{n}$ (in reduced form), then $f(\frac{m}{n}) = \frac{1}{n}$. Find the points where f is continuous.

42. Let M be a metric space, $D \subset M$ dense. Let $f : D \longrightarrow \mathbb{R}$ be a function. Show that if f is uniformly continuous then it has a continuous extension to all of M. (This means that there exists some $g : M \longrightarrow \mathbb{R}$ continuous such that $g(x) = f(x)$ for all $x \in D$.)

43. Let $f : (0, 1) \longrightarrow \mathbb{R}$ be a function. Show that f has a continuous extension to $[0, 1]$ iff it is uniformly continuous.

44. Let K be a compact metric space, $(U_\alpha)_{\alpha \in I}$ an open cover of K. Show that there exists some $\varepsilon > 0$ such that any open ball of radius ε is contained in some U_α.

45. Let M, N be metric spaces, $f : M \longrightarrow N$ be uniformly continuous. If A is a totally bounded subset of M, show that $f(A)$ is also totally bounded.

46. Let M, N be metric spaces, $f : M \longrightarrow N$ a function. Suppose that the restriction of f to any compact subset of M is continuous. Show that f is continuous.

47. If A is a locally compact subspace of a metric space M, show that A is locally closed.

48. Let M be a metric space, A, B be two locally compact subspaces of M. Show that $A \cap B$ is also locally compact.

49. In $\mathbb{R}$ give an example of two locally compact subspaces whose union is not locally compact. Give also an example of a locally compact ssubspace whose complement is not locally compact.

50. Give an example of a locally compact metric space which is not complete.

51. Let M be a proper metric space. Show that M is complete. If A is a relatively compact subset, show that $C(A, \frac{1}{2})$ is compact.

52. If $A \subseteq M$ is connected and $A \subseteq B \subseteq \overline{A}$, show that B is also connected.

53. Suppose $A_1, A_2, \cdots, A_n$ are connected subset of a metric space M with $A_i \cap A_{i+1} \neq \emptyset$ for all $1 \leq 1 < n$. Show that $\cup_{i=1}^n A_i$ is connected.

54. If A, B are connected subsets of M and $\overline{A} \cap B \neq \emptyset$, show that $A \cup B$ is connected.

55. Let M be a metric space, $A, B \in CL(M) - \{\emptyset\}$, $A \cup B$ and $A \cap B$ be connected. Show that both A and B are connected.

56. Let M be a complete metric space, $f : M \longrightarrow M$ a function. Suppose that there is some $c \in (0, 1)$ such that $d(f(x), f(y)) \leq cd(x, y)$ for all $x, y \in M$. Show that there is a unique $x \in M$ such that $f(x) = x$.

57. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = x - \frac{\pi}{2} - \arctan x$$

for all $x \in \mathbb{R}$. Show that $|f(x) - f(y)| < |x - y|$ for all $x, y \in \mathbb{R}$. Show also that f does not have any fixed point. Deduce that $c < 1$ in the previous exercise is essential.

58. Assume $E \subseteq \mathbb{R}^n$. A point $X \in \mathbb{R}^n$ is said to be a condensation point of S if for every $r > 0$ $B(X, r) \cap E$ is uncountable. If E is uncountable, show that there is some $X \in S$ which is a condensation point of E.

59. Let $S, T \subseteq \mathbb{R}^n$. Show that $\overline{S \cap T} \subseteq \overline{S} \cap \overline{T}$ and that $S \cap \overline{T} \subseteq \overline{S \cap T}$. (This is true for general metric spaces as well)

60. Prove that every $F \in CL(\mathbb{R})$ is the intersection of a sequence of open sets.

61. Suppose that $S, T \subseteq \mathbb{R}^n$. Show that $S^\circ \cap T^\circ = (S \cap T)^\circ$ and that $S^\circ \cup T^\circ \subseteq (S \cup T)^\circ$. (This is true for general metric spaces as well)

62. Let $S \subseteq \mathbb{R}^n$. We say that a point $X \in \mathbb{R}^n$ is an accumulation point of S if evry r-ball around X contains some element of S different from X itself. Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^n$. Let $S = \cup_{A \in \mathcal{F}} A$ and $T = \cap_{A \in \mathcal{F}} A$. Prove or disprove:

    (a) If X is an accumulation point of T, then it is an accumulation point of each $A \in \mathcal{F}$.

    (b) If X is an accumulation point of S, then it is an accumulation point of at least one $A \in \mathcal{F}$.

63. Let S be the set of rational numbers in $(0,1)$. Can one write S as the intersection of a countable family of open subsets of $\mathbb{R}$?

64. Let S be a subset of a metric space M, A point $X \in S$ is said to be an isolated point of S if $B(X,r) \cap S = \{X\}$ for some $r > 0$. Now let $S \subseteq \mathbb{R}^n$. Show that the set of isolated points of S is at most countable.

65. Let $\mathcal{B} : \{B((x,x),x) : x \in (0,\infty), x \in \mathbb{Q}\}$. Show that $\mathcal{B}$ is an open cover of $(0,\infty) \times (0,\infty) \subseteq \mathbb{R}^2$.

66. Let $\mathcal{U} : \{(\frac{1}{n}, \frac{2}{n}) : n \in \mathbb{N}, n \geq 2\}$ is an open cover of $(0,1)$ which does not have any finite subcover.

67. Suppose $S \subseteq \mathbb{R}^n$ has the following property: For every $X \in S$ there is some $r > 0$ such that $S \cap B(X,r)$ is countable. Prove that S is countable?

68. Let $S \subseteq \mathbb{R}^n$ be an uncountable subset. Let T be the set of condensation points of S. Prove that

    (a) $S - T$ is at most countable,

    (b) T is closed,

    (c) $S \cap T$ is uncountable, and

    (d) T does not have any isolated points.

69. A set $S \subset \mathbb{R}^n$ is said to be perfect if every point of S is an accumulation point of S. ($S \subseteq \mathbb{R}^n$ is perfect iff it is a closed set whithout isolated points.) Prove that if $F \subseteq \mathbb{R}^n$ is uncountable and closed, then $F = A \cup B$, where A is perfect and B is countable.

70. Let M be a metric space, $A, S \subseteq M$. If $A \subseteq S \subseteq \overline{A}$, we say that A is dense in S. Now let $A, S, T \subseteq M$. If A is dense in S, S is dense in T, show that A is dense in T.

71. Suppose M is a metric space. If $A \subseteq M$ and $B \in OP(M)$ are dense in M, show that $A \cap B$ is also dense in M.

72. Let M be metric space. Assume $K \subseteq L \subseteq M$. Show that K is compact in L iff it is compact in M.

73. Let $a, b \in \mathbb{Q}^c$ with $a < b$, $S = \mathbb{Q} \cap (a,b)$. Show that S is cloed and bounded in $\mathbb{Q}$ and that it is not compact.

74. Let M be a metric space, $A, B \subseteq M$, $\mathcal{F}$ be a family of subsetes of M. The set $\partial(A) = \overline{A} - A^\circ$ is called the boundary of A. Prove that

(a) $A^\circ = M - \overline{(M - A)}$,

(b) $(M - A)^\circ = M - \overline{A}$,

(c) $(A^\circ)^\circ = A^\circ$,

(d) $(\cap_{i=1}^n A_i)^\circ = \cap_{i=1}^n A_i^\circ$,

(e) $(\cap_{A\in\mathcal{F}} A)^\circ \subseteq \cap_{A\in\mathcal{F}} A^\circ$, and that the inclusion can be proper,

(f) $(\cup_{A\in\mathcal{F}} A)^\circ \subseteq \cup_{A\in\mathcal{F}} A^\circ$, and that the inclusion can be proper,

(g) $(\partial(A))^\circ = \emptyset$ if $A \in OP(M)$ or $A \in CL(M)$.

(h) If A is closed and $A^\circ = B^\circ = \emptyset$, then $(A \cup B)^\circ = \emptyset$,

(i) If $\overline{A} \cap \overline{B} = \emptyset$, then $\partial(A \cup B) = \partial(A) \cup \partial(B)$.

75. Let M be metric space. If $x_n \longrightarrow x$ and $y_n \longrightarrow y$ in M, show that $d(x_n y_n) \longrightarrow d(x, y)$.

76. A sequence $(x_n)_n$ in $\mathbb{R}$ satisfies $7x_{n+1} = x_n^3 + 6$ for $n \in \mathbb{N}$. If $x_1 = \frac{1}{2}$, show that $(x_n)_n$ converges and find its limit. What if $x_1 = \frac{5}{2}$?

77. If $x_1 \in (0, 1)$ and $_{n+1} = 1 - \sqrt{1 - x_n}$ for all $n \geq 1$, show that $(x_n)_n$ is decreasing with limit 0. Show also that $\frac{x_{n+1}}{x_n} \longrightarrow \frac{1}{2}$.

78. If $a_{n+2} = \frac{a_n + a_{n+1}}{2}$ for all $n \geq 1$, show that $a_n \longrightarrow \frac{a_1 + 2a_2}{3}$.

79. If $|a_n| < 2$ and $|a_{n+1} - a_n| \leq \frac{1}{8}\left|a_{n+1}^2 - a_n^2\right|$ for all $n \geq 1$, show that $(a_n)_n$ converges.

80. Let $F : \mathbb{N} \longrightarrow \mathbb{R}^d$ be a sequence with $F = (f_1, f_2, \cdots, f_d)$. Show that $F(n) \longrightarrow P \in \mathbb{R}^d$ iff $f_i(n) \longrightarrow d_i$ as $n \longrightarrow \infty$ for each $i = 1, 2, \cdots, d$.

81. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at one point, show that f is continuous and that there is some $a \in \mathbb{R}$ with $f(x) = ax$ for all $x \in \mathbb{R}$.

82. Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \longrightarrow \mathbb{R}$. We say that f is convex if for all $x, y \in [a, b]$ and $\alpha \in [0, 1]$ we have $f((1 - \alpha)x + \alpha y) \leq (1 - \alpha)f(x) + \alpha f(y)$. Show that a convex function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous.

83. Let $S \subseteq \mathbb{R}^n$ be open and connected. Let T be a connected component of $\mathbb{R}^n - S$. Show that $\mathbb{R}^n - T$ is connected.

84. Let M be a metric space, $x \in M$ and $U(x)$ the connected component of M containing x. Show that $U(x) \in CL(M)$.

85. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be linear. Show that T is uniformly continuous.

86. Let E, F be normed vector spaces, $T : E \longrightarrow F$ linear. Show that TFAE:

    (a) T is continuous at one point,

    (b) T is uniformly continuous,

    (c) T maps bounded sets to bounded sets.

87. Let $|\cdot|$ be a norm on $\mathbb{R}^n$. Show that there are $m, M \in (0, \infty)$ such that $m\,|X| \leq \|X\| \leq M\,|X|$ for all $x \in \mathbb{R}^n$. ($\|X\|$ denotes the euclidean norm of $X \in \mathbb{R}^n$.) Hence all norms on $\mathbb{R}^n$ are equivalent.

88. Let K be a compact metric space, $f : K \longrightarrow K$ be a function with

$$d(f(x), f(y)) < d(x, y)$$

    for all $x, y \in K$. Does f necessarily have a fixed point?

89. Let K be a compact metric space. We now know that every continuous complex function on K is bounded. Hence, if $C(K)$ denotes the set of continuous complex functions on K, it is a vector subspace of $\ell^\infty(K)$. Is $C(K)$ closed in $\ell^\infty(K)$.

90. Suppose M is a locally compact separable metric space. Show that there exists a sequence $(K_n)_n$ of compact subsets such that

    (a) $K_n \subset K_{n+1}^\circ$ for all $n \in \mathbb{N}$,

    (b) $M = \cup_n K_n$, and that

    (c) every compact subset of M is contained in at least one of the K_n's.

91. Let E, F be metric spaces, $f : E \longrightarrow F$ a function. The graph of f is defined to be the set $\{(x, f(x)) : x \in E\}$. If f is continuous, show that the graph of f is closed. Suppose that E is compact. Show that f is continuous iff the graph of f is compact.

92. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function with the intermediate value property: If $f(a) < c < f(b)$, there is some x between a and b such that $f(x) = c$. Suppose also that $f^{-1}(\{r\})$ is closed for every $r \in \mathbb{Q}$. Show that f is continuous.

93. Ssuppose that $f : (a, b) \longrightarrow \mathbb{R}$ is continuous and that

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a, b)$. Show that f is convex.

94. Find the limits if they exist:

(a)

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^4+y^4}{x^2+y^2}$$

(b)

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{xy^2}{x^2+y^4}$$

(c)

$$\lim_{\|X\|\longrightarrow\infty} \frac{\|X-X_1\|}{\|X-X_2\|}$$

where X_1, X_2 are given elements of $\mathbb{R}^n$.

95. Show that $\lim_{X\longrightarrow P} f(X) = +\infty$ iff $\lim_{X\longrightarrow P} \frac{1}{f(X)} = 0$ and $f(X) > 0$ for every X in some punctured neighborhood of P.

96. Let $C \subseteq \mathbb{R}^n$ be a closed, convex non-empty subset, $P \in \mathbb{R}^n - C$. Show that there is exactly one point $X \in C$ such that $d(P, C) = d(P, X)$.

97. Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is continuous, that $f(X) > 0$ for all $X \neq 0$ and that $f(cX) - cf(X)$ for all $X \in \mathbb{R}^n$ and $c > 0$. Show that there are $a, b \in (0, \infty)$ such that $a\|X\| \le f(X) \le b\|X\|$ for all $X \in \mathbb{R}^n$.

98. If A, B are two non-empty compact subsets of $\mathbb{R}^n$, define $d(A, B)$ to be the smallest number a with the following property: For every $X \in A$ there is some $Y \in B$ such that $\|X - Y\| \le a$ and for every $Y \in B$ there exists some $X \in A$ such that $\|X - Y\| \le a$. Show that d is a metric on the space $\mathcal{K}(\mathbb{R}^n)$ of all non-empty compact subsets of $\mathbb{R}^n$. Is this space complete? If $\mathcal{KC}(\mathbb{R}^n)$ denotes the set of elements of $\mathcal{K}(\mathbb{R}^n)$ which are also convex, is it closed?

99. Suppose that $K \subset \mathbb{R}^n$ is compact, convex, symmetric about 0 and that K contains a euclidean neighborhood of 0. Let $|0| = 0$ and define, for $X \neq 0$,

$$|X| := \frac{1}{\max\{t : tX \in K\}}.$$

Show that $X \mapsto |X|$ is a norm on $\mathbb{R}^n$.

100. Let M be a metric space, $x \in M$, and $f, g : M \longrightarrow \mathbb{C}$ two functions. If f is continuous at x, and $\lim_{y \longrightarrow x} g(y) = 0$, show that

$$\lim_{y \longrightarrow x} f(y)g(y) = 0.$$

101. Suppose f is differentiable in (a, b).

    (a) If $f'(x) \geq 0$ for all $x \in (a, b)$, show that f is monotonically increasing.

    (b) If $f'(x) \leq 0$ for all $x \in (a, b)$, show that f is monotonically decreasing.

    (c) If $f'(x) = 0$ for all $x \in (a, b)$, show that f is constant.

102. Let $f : (a, b) \longrightarrow \mathbb{C}$ be a function, $x \in (a, b)$. We can define the derivative of f at x in the same way as the real-valued case. We could also write $f = u + iv$ where u, v are real valued functions. Show that then f is differentiable iff both u and v are differentiable and in this case we have

$$f'(x) = u'(x) + iv'(x).$$

103. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Show that f is constant.

104. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function withe a bounded derivative. Let $\varepsilon > 0$ and define $f(x) = x + \varepsilon g(x)$ for all $x \in \mathbb{R}$. Prove that f is a one-to-one function if ε is small enough.

105. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0,$$

where $C_0, C_1, \cdots, C - n$ are real constants, prove that the equation

$$C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n = 0$$

has at least one root between 0 and 1.

106. Suppose $f : (0, \infty) \longrightarrow \mathbb{R}$ is differentiable and $f'(x) \longrightarrow 0$ as $x \longrightarrow \infty$. Show that $f(x + 1) - f(x) \longrightarrow 0$ as $x \longrightarrow \infty$.

107. Suppose

(a) f is continuous for $x \geq 0$,

(b) $f'(x)$ exists for $x > 0$,

(c) $f(0) = 0$,

(d) f' is monotonically incerasing.

If $g(x) = \frac{f(x)}{x}$ for all $x > 0$, show that g is monotonically increasing.

108. Suppose $f'(x)$ and $g'(x)$ exist, $g'(x) \neq 0$, $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

Show also that this holds for complex functions as well.

109. Let f be a continuous real function. Suppose that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \longrightarrow 3$ as $x \longrightarrow 0$. Does $f'(0)$ exist?

110. Suppose f and g are complex differentiable functions on $(0, 1)$,

$$f(x) \longrightarrow 0, \; g(x) \longrightarrow 0, \; f'(x) \longrightarrow A \text{ and } g'(x) \longrightarrow B$$

as $x \longrightarrow 0$. ($B \neq 0$) Show that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

111. Suppose f is defined in a neighborhood of x and suppose that $f''(x)$ exists. Show that

$$\lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show that the limit may exist although even if $f''(x)$ does not.

112. If $f(x) = |x|^3$, compute $f'(x)$ and $f''(x)$ for all real x and show that $f'''(0)$ does not exist.

113. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Prove the following statements:

(a) f is continuous iff $a > 0$.

(b) $f'(0)$ exists iff $a > 1$.

(c) f' is bounded iff $a > 1 + c$.

(d) $f''(0)$ exists iff $a > 2 + c$.

(e) f'' is bounded iff $a \leq 2 + 2c$.

(f) f'' is continuous iff $a > 2 + 2c$.

114. Suppose $a \in \mathbb{R}$, f is a twice-differentiable function on (a, ∞), and M_0 , M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞). Prove that

$$M_1^2 \leq 4M_0M_2.$$

115. Suppose f is a real, twice-differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \qquad f(0) = 0, \qquad f(1) = 1, \qquad f'(0) = 0.$$

Prove that $f'''(x) \geq 3$ for some $x \in (-1, 1)$.

116. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function.

(a) If f is differentiable and $f(t) \neq 1$ for all $t \in \mathbb{R}$, show that f can have at most one fixed point.

(b) Show that the function $f(t) = t + (1 + e^t)^{-1}$ has no fixed point, although $0 < f'(t) < 1$ for all real t.

(c) If there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, show that f must have a fixed point.

117. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A\,|f(x)|$ for all $x \in [a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

118. Let $f : (a, b) \longrightarrow \mathbb{R}$ be differentiable with a bounded derivative. Show that f is uniformly continuous.

119. Suppose $f : (a, \infty) \longrightarrow \mathbb{R}$is differentiable ($a \in \mathbb{R}$), and $\lim_{x\to\infty} f'(x) = \infty$. Show that f is not uniformly continuous.

120. Suppose $f : (a, \infty) \longrightarrow \mathbb{R}$ is differentiable. If $\lim_{x\to\infty} f'(x) = g$, show that $\lim_{x\to\infty} \frac{f(x)}{x} = g$ as well.

121. Suppose $f : (0,1]$ is differentiable with $|f'(x)| < 1$ for all $x \in (0,1]$. Define $a_n = f(\frac{1}{n})$ for all $n \in \mathbb{N}$. Show that $(a_n)_n$ converges.

122. Suppose $f : (a,b) \longrightarrow \mathbb{R}$ is differentiable and $c \in (a,b)$. Assume that $\lim_{x\to c} f'(x)$ exists. Show that $\lim_{x\to c} f'(x) = f'(c)$.

123. Suppose $f : (a,b) \longrightarrow \mathbb{R}$ is continuous and is differentiable except possibly at $c \in (a,b)$. If $f'(x) \longrightarrow A$ as $x \longrightarrow c$, show that f is also differentiable at c and $f'(c) = A$.

124. For each $n \in \mathbb{N}$, let $g_n : [0,1] \longrightarrow \mathbb{R}$ be an integrable function. Define $G_n(x) = \int_0^x g_n(t)dt$ for all $n \in \mathbb{N}$ and $x \in [0,1]$. Show that $(G_n)_n$ has a uniformly convergent subsequence.

125. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

(a) Show that f is smooth.

(b) Is f analytic?

(c) Define
$$g(x) = e^2 f(1-x)f(x+1).$$
Show that g is smooth, identitaclly zero outside $(-1,1)$, positive on $(-1,1)$, and takes the value 1 at $x - 0$.

(d) Show that $g(x) = e^{-2x^2/(x^2-1)}$ for all $|x| < 1$.

126. Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions for each $n \in \mathbb{N}$ with $f_n(0) = 0$ and $|f_n'(x)| \leq 2$ for all n, x. If $f_n \longrightarrow g$ pointwise, show that g is continuous.

127. Suppose $(f_n)_n$ is a sequence of differentiable real functions on a compact interval $[a,b]$ such that $|f_n(x)| \leq M$ and $|f_n'(x)| \leq M$ for all n, x. ($M \in \mathbb{R}$ is fixed.) Show that then $(f_n)_n$ has a uniformly convergent subsequence.

128. Let (f_n) be a sequence of functions defined on an open interval I satisfying $|f_n(x)| \leq F(x)$ and $|f_n'(x)| \leq G(x)$ for all n, x, where $F, G : I \longrightarrow \mathbb{R}$ are constinuous functions. Prove that (f_n) has a subsequence which converges uniformly on every compact subset of I.

129. Prove that the set of polynomials of degree$\leq N$ with coefficients in $[-1,1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.

130. Prove that the family of polynomials $P(x)$ of degree $\leq N$ satisfying $|P(x)| \leq 1$ on $[0,1]$ is uniformly equicontinuous on $[0,1]$.

131. If $(f_n)_n$ is a uniformly equicontinuous sequence of functions on a compact interval and $f_n \longrightarrow f$ pointwise, prove that $f_n \longrightarrow f$ uniformly.

132. Let F be finite set of continuous functions on a compact interval. Show that $\mathcal{F} := \{\sum_{f\in F} a_f f : |a_f| \leq 1 \text{ for all } f \in F\}$ is uniformly bounded and uniformly equicontinuous.

133. Let $(f_n)_n$ be a sequence of uniformly bounded uniformly equicontinuous functions on a bounded open interval (a,b). Show that the functions can be extended to the compact interval $[a,b]$ so that they are still uniformly bounded and uniformly equicontinuous.

134. Give an example of a sequence of functions that is unformly equicontinuous but not uniformly bounded.

135. Give an example of a sequence of real functions on $\mathbb{R}$ that is uniformly bounded and uniformly equicontinuous but doesn't have any uniformly convergent subsequence.

136. Prove that the sequence $f_n(x) = \sin nx$ is not uniformly equicontinuous on any non-trivial compact interval.

137. Suppose that $(f_n)_n$ is a sequence of functions on a compact interval that is pointwise bounded and pointwise equicontinuous. Show that it has a subsequence which converges pointwise.

138. Suppose that $f : [0,1] \longrightarrow \mathbb{R}$ is continuous. If f satsifies

$$\int_0^x f(t)dt = x\sin x + \int_0^x \frac{f(t)}{1+t^2}dt,$$

find f.

139. Find

$$\lim_{x\to 3} \frac{x}{x-3}\int_0^x \frac{\sin t}{t}dt.$$

140. Find tha maximum value of the function $x+y$ on the unit circle $\{(x,y) : x^2+y^2=1\}$.

141. Find the largest area of a rectangle inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(Here $a, b > 0$)

142. Let $A = \{(x, y) : y^2 = 2x\}$. If $p = (1, 4) \in \mathbb{R}^2$, find $d(p, A)$.

143. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous. Show that

$$\lim_{p \longrightarrow \infty} \left(\int_a^b |f(x)|^p \, dx \right)^{1/p} = \|f\|_\infty .$$

144. Show that, for $m, n \in \mathbb{N}$, we have

$$\int_{-\pi}^{\pi} \sin mx \cos nx dx = 0.$$

145. Let $m, n \in \mathbb{N}$ with $m \neq n$. Show that

$$\int_{-\pi}^{\pi} \sin mx \sin nx dx = \int_{-\pi}^{\pi} \cos mx \cos nx dx = 0.$$

146. Suppose that $f : [0, 1] \longrightarrow \mathbb{R}$ is continuous. Prove that

$$\int_0^{\pi} x f(\sin x) dx = \frac{1}{2}\pi \int_0^{\pi} f(\sin x) dx.$$

147. Suppose $a, b \in \mathbb{R}$ with $|a| \neq |b|$. Prove that

$$\lim_{x \to 0} \frac{1}{x} \int_0^x \sin at \cos bt dt = 0.$$

148. Let f be a continuous function on $[a, b]$. Suppose there exists a constant K such that

$$|f(x)| \leq K \int_a^x |f(t)| \, dt$$

for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

149. Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is twice continuously differentiable. Show that

$$\int_a^b x f''(x) dx = bf'(b) - f(b) + f(a) - af'(a).$$

150. Let $m, n \in \mathbb{N}$. Show that

$$\int_0^1 x^m(1-x)^n dx = \int_0^1 x^n(1-x)^m dx$$

.

151. For $f, g \in C([a,b])$ define $d_1(f,g) = \int_a^b |f(x) - g(x)|\, dx$. Show that d_1 is a metric on $C([a,b])$.

152. Prove that

$$\lim_{h\to 0} \int_{-a}^{a} \frac{h}{h^2+x^2} dx = \pi.$$

153. Prove that, if $f : [-1,1] \longrightarrow \mathbb{R}$ is continuous, then

$$\lim_{h\to 0} \int_{-1}^{1} \frac{h}{h^2+x^2} f(x) dx = \pi f(0).$$