Math 112 Final

June 7, 2004

1 Binary Tree

Definition 1.1 *Binary tree* is the partial order relation $(T, <)$ satisfying:
i) there is $r \in T$ such that $r \leqslant x$ for all $x \in T$;
ii) if $x \in T$, then $\{y : y < x\}$ is finite and linearly ordered by $<$;
iii) if $x \in T$, then there is a finite set $\{y, z\}$ consisting of exactly two incomparable elements such that $y, z > x$ and if $t > x$ then $t \geqslant y$ or $t \geqslant z$.

Definition 1.2 A branch $\mathcal{B}$ in the *binary tree* is a totally ordered subset satisfying:
if $x \in \mathcal{B}$ and $y < x$ then $y \in \mathcal{B}$.

Exercise 1.1 Show that there are uncountably many branches in the *binary tree.*

Exercise 1.2 Show that $\mathbb{N}$ has uncountably many infinite subsets such that their two by two intersections are all finite.

Exercise 1.3 Note that we defined the *binary tree* as it is a unique object. If we remove the first condition from its definition, is it still a unique object?

2 Orders

Definition 2.1 A total order $(T, <)$ is called *discrete* if for any $x \in T$ there is a unique $s \in T$ and a unique $p \in T$ such that $p < x < s$ and for any $y < x < z$ we have $y \leqslant p$ and $z \geqslant s$.

Exercise 2.1 Find two non-isomorphic countable discrete orders.

Definition 2.2 A total order $(T, <)$ is called *dense* if for any $x, y \in T$ with $x < y$, there is a $z \in T$ such that $x < z < y$.

Exercise 2.2 Find two non-isomorphic countable dense orders.

Exercise 2.3 Find an uncountable discrete order.

Exercise 2.4 Find an uncountable dense order.

Exercise 2.5 Find an uncountable order which is neither dense or discrete or a well-order.

Exercise 2.6 Show that a discrete order and a dense order cannot be isomorphic.

Exercise 2.7 Can there be a well-order which is also dense?

Exercise 2.8 Show that there are uncountably many non-isomorphic countable well-orders.

Exercise 2.9 Show that there are uncountably many non-isomorphic countable discrete orders without a first point and a last point.