Set Theory (Math 112)
First Midterm

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1 Binary Tree

A **binary tree** is a partial ordered set $(T, <)$ such that

i) T has a smallest element, i.e. there is an $r \in T$ such that $r \leq x$ for all $x \in T$.

ii) For all $x \in T$, the set $\{y : y < x\}$ of predecessors of x is finite and linearly ordered by $<$.

iii) For all $x \in T$, there are exactly two immediate successors of x, i.e. there are two elements $y \neq z$ such that y, $z > x$ and if $t > x$ then either $t \geq y$ or $t \geq z$.

A **branch** $\mathcal{B}$ in a binary tree is a totally ordered subset such that for all $x \in \mathcal{B}$, if $y < x$ then $y \in \mathcal{B}$.

Ia. Show that the set of elements that have exactly n predecessors is finite. Show that this set has exactly 2^n elements. Conclude that a binary tree is countably infinite.

Ib. Show that any two distinct branches intersect in a finite set.

Ic. Show that any two binary trees are isomorphic.

Id. Show that there is a one to one correspondence between a binary tree and the set of finite sequences of zeroes and ones.

Ie. Show that there are uncountably many branches in a binary tree.

If. Conclude that $\mathbb{N}$ has uncountably many infinite subsets whose two by two intersections are all finite.

2 Orders

A total order $(T, <)$ is called **discrete** if any of its elements has a unique immediate predecessor and a unique immediate successor, i.e. if for any $x \in T$ there are unique $s, p \in T$ such that $p < x < s$ and for any $y < x < z$ we have $y \leq p$ and $z \geq s$.

IIa. Find two non-isomorphic countable discrete orders.

IIb. Find an uncountable discrete order.

A total order $(T, <)$ is called **dense** if for any $x, y \in T$ with $x < y$, there is a $z \in T$ such that $x < z < y$. A total order is said to be **without end points** if it has neither first nor last element.

IIc. Show that a discrete order and a dense order cannot be isomorphic.

IId. Find an uncountable dense order without end points.

IIe. Find two non-isomorphic countable dense orders.

IIf. Find an uncountable order which is neither dense or discrete or a well-order.

IIg. Show that there are uncountably many non-isomorphic countable well-orders.

IIh. Show that any two countable dense orders without end points are isomorphic.

IIi. Find two nonisomorphic uncountable dense order without end points.