Math 151 : Analysis 1
Sample Questions

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Note: These are just some problems for self study. More problems you can find in the script of Ali Nesin on Analysis 1. Some of these problems are already from that script.

Problems

1. Show that $||x| - |y|| \leq |x - y|$ for any $x, y \in \mathbb{C}$.
2. Find and prove (by induction) formulas for the sums
$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \cdots + \frac{2n+1}{n^2 \cdot (n+1)^2}$$
and
$$1 + 9 + 25 + \cdots + (2n+1)^2.$$
3. Prove by induction that
$$(a^n + a^{n-1}b + a^{n-2}b^2 + \cdots ab^{n-1} + b^n)(a - b) = a^{n+1} - b^{n+1}.$$
4. Find supremum and infimum (if they exist) of the following sets:
   (a) $\{x \in \mathbb{R} : x^2 + x - 1 < 0\}$,
   (b) $\{x \in \mathbb{R} : x^2 + x - 1 > 0\}$,
   (c) $\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}, n > 0\}$,
   (d) $\{\frac{1}{1-x} : x \in \mathbb{R}, x > 1\}$,
   (e) $\{\frac{1}{1-x} : x \in \mathbb{R}, x < 1\}$.
5. For a natural number n we define $\nu(n) = \sup\{m \in \mathbb{N} : 2^m \leq n\}$. Show that $\lim_{n\to\infty} \nu(n)/n = 0$.

6. Show that if a, $b > 0$ then $\lim_{n\to\infty}(a^n + b^n)^{\frac{1}{n}} = \max(a, b)$.

7. Let $f : [a, b] \longrightarrow [a, b]$ be a function with the following property: There exists a real number $c \in (0, 1)$ such that for any $x, y \in [a, b]$ one has

$$|f(x) - f(y)| < c|x - y|.$$

   (a) Show that f is continuous, and

   (b) Show that there exists an $x \in [a, b]$ such that $f(x) = x$.

8. Let $(x_n)_n$ be a convergent sequence in $\mathbb{C}$ (or a Banach space X), $r > 0$ and assume that $|x_n| \leq r$ for all n. Show that $|\lim_{n\to\infty} x_n| \leq r$.

9. Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function with the following properties:

   (a) $f(1) = a > 0$,

   (b) $f(x + y) = f(x) \cdot f(y)$ for all $x, y \in \mathbb{R}$.

   Show that $f(t) = a^t$ for all $t \in \mathbb{R}$.

10. Show that there does not exist any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which assumes every $x \in \mathbb{R}$ twice.

11. Does there exist a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which assumes every real number three times? (Yes!)

12. Assume $f, g : [0, 1] \longrightarrow [0, 1]$ be continuous functions and assume that $f(0) \geq g(0)$ and $f(1) \leq g(1)$. Show that there exists an $x \in [0, 1]$ such that $f(x) = g(x)$.

13. Assume that f is a continuous real-valued function on $[0, 1]$ and that $f(0) = f(1)$. If $n \in \mathbb{N}$ is positive show that there is a point x in $[0, 1]$ such that $f(x) = f(x + \frac{1}{n})$.

14. Assume that X is a complete metric space which is also connected.Show that for any $a \in X$ and any $r \geq 0$ there is a $b \in X$ such that $d(a, b) = r$. Conclude that if x has more than one point, show that X must be uncountable.

15. Show that the set of all sequences in $\mathbb{N}$ is uncountable. Show that the set of all finite subsets of $\mathbb{N}$ is countable.