1. Let \( A \) be an abelian group. Let \( B := \{ a \in A : a \) has finite order\}. 

1a. Show that \( B \) is a subgroup of \( A \).

Recall that \( a \in A \) has finite order if and only if \( a^n = 1 \) for some natural number \( n > 0 \). Thus, 

\[
B := \{ a \in A : a^n = 1 \text{ for some natural number } n > 0 \}
\]

Since \( 1^1 = 1 \), \( 1 \) is in \( B \).

Assume \( a \) is in \( B \). We want to show that \( a^{-1} \) is also in \( B \). Since \( a \) is in \( B \), \( a^n = 1 \) for some \( n > 0 \). Since \( (a^{-1})^n = (a^n)^{-1} = 1^{-1} = 1 \) (the first equality has been proved in class), \( a^{-1} \) is also in \( B \).

Assume \( a_1, a_2 \in B \). We want to show that \( a_1a_2 \in B \). Since \( a_1 \) and \( a_2 \) are in \( B \), there are positive integers \( n_1 \) and \( n_2 \) such that \( a_1^{n_1} \) and \( a_2^{n_2} = 1 \). Now,

\[
(a_1a_2)^{n_1n_2} = a_1^{n_1n_2}a_2^{n_1n_2} = (a_1^{n_1})^{n_2}(a_2^{n_2})^{n_1} = 1^{n_2}1^{n_1} = 1 \cdot 1 = 1.
\]

Therefore \( a_1a_2 \) has finite order and it is in \( B \).

1b. Show that in the group \( A/B \), the order of every nonidentity element is infinite.

Let \( aB \in A/B \) be an element of finite order. We want to show that \( aB \) is the identity element of the group \( A/B \), i.e. we want to show that \( aB = B \), i.e. we want to show that \( a \in B \). Since \( aB \) has finite order, \( (aB)^n \) is equal to the identity element of \( A/B \), which is \( B \). So, \( (aB)^n = B \). But \( (aB)^n = a^nB \). Hence \( a^nB = B \). This means that \( a^n \in B \). Note that, till now, we did not use the definition of \( B \). Since \( a^n \in B \), from the definition of \( B \), it follows that \( (a^n)^m = 1 \) for some positive integer \( m \). Hence \( a^{nm} = 1 \) and \( a \) has finite order. Therefore \( a \in B \).

2. We consider \( \mathbb{R} \) as a group under addition. Since \( \mathbb{Z} \leq \mathbb{R} \), we can consider the group \( G := \mathbb{R}/\mathbb{Z} \).

2a. Show that every element of \( G \) can be written as \( r + \mathbb{Z} \) for some unique \( r \in \mathbb{R} \) with \( 0 \leq r < 1 \).
Let $a + Z$ be an element of $\mathbb{R}/\mathbb{Z}$. We can write
\[ a = r + z \]
for some $z \in \mathbb{Z}$ and $r \in [0, 1)$. For example, if $a = 3.14$, we can take $z = 3$ and $r = 0.14$. If $a = -5.4$, we can take $z = -6$ and $r = 0.6$. In general $z$ is equal to the integer part $[a]$ of $a$ and $r = a - [a]$. Now $a + Z = r + z + Z = r + Z$. This shows that every element of $\mathbb{R}/\mathbb{Z}$ can be written as $r + Z$ for some $r \in [0, 1)$.

We now show the uniqueness. Assume that for $0 \leq r, s < 1$, $r + Z = s + Z$. We want to show that $r = s$. Since $r + Z = s + Z$, $r - s \in \mathbb{Z}$. But $r - s \in (-1, 1)$. Therefore $r - s = 0$ and $r = s$.

2b. Show that if $q \in \mathbb{Q}$, then $q + Z$ is an element of finite order of $G$.

Let $q = m/n$. Then $n(q + Z) = nq + Z = m + Z = Z$. Therefore the order of $q + Z$ is finite (it divides $n$).

2c. Find all elements of order 2, 3 and 6 of $G$.

Let $r + Z$ be an element of order 6 of $G$. By part (a), we may assume that $r \in [0, 1)$. Since the order of the element $r + Z$ of $\mathbb{R}/\mathbb{Z}$ is 6,
\[ Z = 6(r + Z) = 6r + Z. \]
Therefore, $6r \in \mathbb{Z}$ and $r$ is equal to one of $0/6, 1/6, 2/6, 3/6, 4/6, 5/6$. But it is easy to see that
\[
\begin{align*}
0/6 + Z & \text{ has order 1} \\
2/6 + Z & \text{ has order 3} \\
3/6 + Z & \text{ has order 2} \\
4/6 + Z & \text{ has order 3}
\end{align*}
\]
Therefore only $1/6 + Z$ and $5/6 + Z$ have order 6.

Similarly, the only element of order 2 is $1/2 + Z$ and the elements of order 3 are $1/3 + Z$ and $2/3 + Z$.

2d. For a fixed integer $n > 0$, find all elements of order $n$ of $G$.

The calculations above suggest and one can prove in a similar way that the set of elements of order $n$ is $\{n/m + Z : n$ and $m$ are prime to each other\}.

3. Let $G$ be a group and let $H \triangleleft G$ be a normal subgroup of $G$.

3a. Show that $C_G(H) := \{g \in G : gh = hg \text{ for all } h \in H\}$ is a normal subgroup of $G$.

We have seen in the first midterm that $C_G(H)$ is a subgroup of $G$ for any subset $H$ of $G$. Thus we only need to prove that $C_G(H)$ is normal in $G$. We have to show that for any $g \in G$,
\[ g^{-1}C_G(H)g \subseteq C_G(H). \]
Let \( c \in C_G(H) \). Thus \( c \) commutes with every element of \( H \). We want to show that
\[
g^{-1}cg \in C_G(H).
\]
Thus we want to show that \( g^{-1}cg \) commutes with every element of \( H \). Accordingly, let \( h \) be any element of \( H \) and try to show that \( g^{-1}cg \) and \( h \) commute with each other, i.e. that
\[
(1) \quad g^{-1}cg \cdot h = h \cdot g^{-1}cg.
\]
We know that \( ghg^{-1} \in H \), because \( H \) is normal in \( G \). Since \( ghg^{-1} \in H \) and since \( c \) commutes with every element of \( H \), \( c \) commutes with \( ghg^{-1} \). Thus
\[
c \cdot ghg^{-1} = ghg^{-1} \cdot c.
\]
(1) follows from this easily.

3b. For \( x \in G \), define \( B(x) := \{g \in G : g^{-1}x^{-1}gx \in H\} \). Show that \( B(x) \) is a subgroup of \( G \) that contains \( H \).

Since \( 1^{-1}x^{-1}1 = 1 \in H \), \( 1 \) is in \( B(x) \).
Let \( g \) be an element of \( B(x) \). Thus
\[
(2) \quad g^{-1}x^{-1}gx \in H
\]
We want to show that \( g^{-1} \) is also in \( B(x) \), i.e. that
\[
gx^{-1}g^{-1}x \in H.
\]
By (2), \( (g^{-1}x^{-1}gx)^{-1} \in H \), i.e. \( x^{-1}g^{-1}xg \in H \). Now using the fact that \( H \) is normal in \( G \), we get \( g(x^{-1}g^{-1}xg)g^{-1} \in H \), i.e. \( gx^{-1}g^{-1}x \in H \). This is what we wanted to prove.

Let \( g_1, g_2 \) be two elements of \( B(x) \). Thus
\[
(3) \quad g_1^{-1}x^{-1}g_1x \in H \quad \text{and} \quad g_2^{-1}x^{-1}g_2x \in H.
\]
We want to show that \( g_1g_2 \in B(x) \), i.e. that \( (g_1g_2)^{-1}x^{-1}(g_1g_2)x \in H \), i.e. that
\[
(4) \quad g_2^{-1}g_1^{-1}x^{-1}g_1g_2x \in H.
\]
Since
\[
g_2^{-1}g_1^{-1}x^{-1}g_1g_2x = g_2^{-1}(g_1^{-1}x^{-1}g_1x)g_2 \cdot (g_2^{-1}x^{-1}g_2x),
\]
by formulas in (3) and by the fact that \( H \) is normal in \( G \), (4) follows.