First Midterm
Math 120B (Fall 1994)

Ali Nesin

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Math 120 A
Correction of the First Midterm
Ali Nesin
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1. *Let G be a group.*

1a. *For $a \in G$, define $C_G(a) = \{g \in G : ga = ag\}$. Show that $C_G(a)$ is a subgroup of G.*

We first show that the subset $C_G(a)$ is closed under the product. Let $g, h \in C_G(a)$. Therefore we know that

$$ga = ag \tag{1}$$

and

$$ha = ah. \tag{2}$$

We want to show that $gh \in C_G(a)$, i.e. that $(gh)a = a(gh)$. We compute directly by using associativity and equations (1) and (2):

$$(gh)a = g(ha) \overset{(2)}{=} g(ah) = (ga)h \overset{(1)}{=} (ag)h = a(gh).$$

Therefore $(gh)a = a(gh)$ and $gh \in C_G(a)$.

We next show that $C_G(a)$ is closed under inversion, i.e. that if $g \in C_G(a)$, then $g^{-1} \in C_G(a)$. Let $g \in C_G(a)$. Therefore (1) holds. Multiply both sides of (1) by g^{-1} to obtain $ag^{-1} = g^{-1}a$. Therefore $g^{-1} \in C_G(a)$.

Finally we show that $1 \in C_G(a)$. This is trivial, because $1a = a = a1$.

All these show that $C_G(a)$ is a subgroup.

1b. *For $A \subseteq G$, define $C_G(A) = \{g \in G : ga = ag \text{ for all } a \in A\}$. Show that $C_G(A)$ is a subgroup of G.*

Note that an element g is in $C_G(A)$ if and only if g is in $C_G(a)$ for all $a \in A$. Thus

$$C_G(A) = \cap_{a \in A} C_G(a).$$

Since, by part (a), each $C_G(a)$ is a subgroup and since intersection of subgroups is a subgroup, $C_G(A)$ is also a subgroup.

1c. *Let $Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}$. Show that $Z(G)$ is a subgroup of G.*

Clearly $Z(G) = C_G(G)$. Therefore $Z(G)$ is a subgroup by part (b).

2. *Let $\phi : G \longrightarrow H$ be a homomorphism between two groups G and H.*

2a. *Show that $\phi(1) = 1$.*

We compute directly: $\phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1)$. Simplifying $\phi(1)$ from both sides, we get $\phi(1) = 1$.

2b. *Show that $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G$.*
Using part (2a), we compute:

$$1 \overset{(2a)}{=} \phi(1) = \phi(xx^{-1}) = \phi(x)\phi(x^{-1}).$$

Multiplying both sides of the equality above by $\phi(x)^{-1}$ from the left, we get

$$\phi(x)^{-1} = \phi(x^{-1}).$$

2c. *Show that the subset $\{g \in G : \phi(g) = 1\}$ of G is a subgroup of G.*
Let $K = \{g \in G : \phi(g) = 1\}$. We want to show that K is a subgroup of G.

We first show that K is closed under the multiplication of G. Let $g, h \in K$. Thus $\phi(g) = 1 = \phi(h)$. We want to show that $gh \in K$, i.e. that $\phi(gh) = 1$. We compute: $\phi(gh) = \phi(g) \cdot \phi(h) = 1 \cdot 1 = 1$.

Next we show that K is closed under inversion. Let $g \in K$. Thus $\phi(g) = 1$. We want to show that $g^{-1} \in K$, i.e. that $\phi(g^{-1}) = 1$. We compute using part (2b):

$$\phi(g^{-1}) \overset{(2b)}{=} \phi(g)^{-1} = 1^{-1} = 1.$$

Finally we show that $1 \in K$. This is part (a): $\phi(1) = 1$.

These show that K is a subgroup of G.

2d. *Show that the image of ϕ is a subgroup of H. (Recall that the image of ϕ is the set $\phi(G) = \{h \in H : h = \phi(g)$ for some $g \in G\}$.*

We first show that $\phi(G)$ is closed under the multiplication of H. Let $h_1, h_2 \in \phi(G)$. Thus

$$h_1 = \phi(g_1) \quad \text{and} \quad h_2 = \phi(g_2) \tag{3}$$

for some $g_1, g_2 \in G$. We want to show that $h_1h_2 \in \phi(G)$, i.e. that $h_1h_2 = \phi(g)$ for some $g \in G$. As the following computation will show it is enough to take $g = g_1g_2$:

$$h_1h_2 \overset{(3)}{=} \phi(g_1)\phi(g_2) = \phi(g_1g_2).$$

We next show that $\phi(G)$ is closed under inversion. Let $h \in \phi(G)$. Thus $h = \phi(g)$ for some $g \in G$. We want to show that $h^{-1} \in \phi(G)$. We compute using part (2b):

$$h^{-1} = \phi(g)^{-1} \overset{(2b)}{=} \phi(g^{-1}).$$

Finally we show that $1 \in \phi(G)$. We need to show that $1 = \phi(g)$ for some $g \in G$. But, by part (2a), $1 = \phi(1)$. Therefore we can take $g = 1$.

All these show that $\phi(G)$ is a subgroup of G.