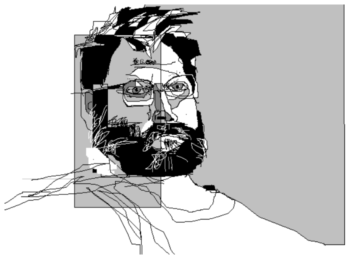
Algebra (Math 211)



Resit
July 8th, 2000

Make full sentences with correct punctuation. Explain your ideas. Although the maximal grade one can get is 150, it is enough to collect 50 points to pass. Be aware that the number of points of each question is supposed to reflect more of less its difficulty.

1a. Show that if the greatest common divisor of two integers n and m is d then there are integers x and y such that $ax + by = d$ (10 pts.)

1b. Find x and y if $a = 1980$ and $b = 714$. (5 pts.)

1c. Does the converse of part a hold? Prove or disprove. (5 pts.)

2. Let R be a commutative ring with 1. Assume that the set of noninvertible elements of R is an ideal I of R. Show that I is necessarily a maximal ideal of R and that it is the only maximal ideal of R. (10 pts.)

3a. Show that the only ideals of a field[1] F are $\{0\}$ and F. (3 pts.)

3b. Conversely, show that a commutative ring with has only two ideals is a field. (7 pts.)

4. Let p be a prime and $R = \{a/b : a, b \in \mathbb{Z}, b \neq 0$ and p does not divide $b\}$.

4a. Show that R is a subring of $\mathbb{Q}$. (4 pts.)

4b. Find invertible elements of R. (7 pts.)

4c. Show that the set of noninvertible elements of R form an ideal, say $\mathfrak{m}$, of R. (5 pts.)

4d. Show that $R/\mathfrak{m}$ is a field. (7 pts.)

5. Let K and L be two fields.

5a. Show that the product ring structure $K \times L$ is not a field. (3 pts.)

5b. Show that $K \times L$ has only 4 ideals. (8 pts.)

5c. Find all the maximal ideals of $(\mathbb{Z}/2\mathbb{Z})^n$. (6 pts.)

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[1] Commutative rings with 1 where every nonzero element is invertible.

6. Show that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is a prime. (6 pts.)

7. Find a maximal ideal of

7a. $\mathbb{Z}[X, Y]$ (3 pts.)

7b. $\mathbb{Z}[X]/\langle X^2 - 1\rangle$. (4 pts.)

7c $\mathbb{Z}[X]/\langle X^2 - 30\rangle$. (6 pts.)

8. Find the set of invertible elements of

8a. $\mathbb{Z}[X, Y]$ (3 pts.)

8b. $\mathbb{Q}[X]/\langle X^2 - 1\rangle$. (5 pts.)

8c. $\mathbb{Z}[X]/\langle X^2 - 1\rangle$. (14 pts.)

8d. $\mathbb{Z}[X, Y]/\langle XY - 1\rangle$ (29 pts.)