Final
Math 120B (Fall 1994)

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It is better to answer a few questions correctly than to give partial answers to lots of them. Therefore, try to concentrate your efforts to a few problems. The first two problems are computational in nature and I believe they are easier than the rest. Think calmly with a clear head and enjoy yourself.

1a. Find all elements of order 2 of Sym(4).
1b. Is the subset $\{g \in \mathrm{Sym}(4) : g^2 = 1\}$ a subgroup of Sym(4)?

2. Let $G := \{\begin{pmatrix} x & y \\ z & t \end{pmatrix} : x, y, z, t \in \mathbb{R}$ and $xt - yz = 1\}$.

2a. Is G a group? Why? What is the inverse of the element $\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ of G.

2b. Let $H := \{\begin{pmatrix} x & y \\ z & t \end{pmatrix} : x, y, z, t \in \mathbb{Z}$ and $xt - yz = 1\}$. H is a subgroup of G. Is H a normal subgroup of G? (Justify your answer).
2c. Find all elements of order 2 of G.
2d. Show that the subset $Z := \{g \in G : g^2 = \mathrm{Id}\}$ is a normal subgroup of G. (Recall that Id is the identity matrix).
2e. Find all elements of order 2 of G/Z.

3. Let G be a group. Assume that the map $\phi : G \longrightarrow G$ defined by $\phi(g) = g^{-1}$ is a homomorphism of G. Show that G is abelian.

4. Let G be a group and let A be a normal subgroup of G. Assume that G/A is abelian. Show that $x^{-1}y^{-1}xy \in A$ for all $x, y \in G$.

5. Let G be a group. Let A be a subgroup of G. Assume that for all $x, y \in G$, $x^{-1}y^{-1}xy \in A$.
5a. Show that A is a normal subgroup of G.
5b. Show that G/A is an abelian group.

6. Let H be a subgroup of index 2 of a group G. Show that H is a normal subgroup of G.