1. **How many abelian groups are there up to isomorphism of order 67500?**

   **Answer:** Since $67500 = 675 \times 10^2 = 25 \times 27 \times 10^2 = 2^2 \times 3^2 \times 5^4$, the answer is $2 \times 2 \times 5 = 20$

   For the 2-part of the group we have two choices: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$.

   For the 3-part of the group we have two choices: $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z}$.

   For the 5-part of the group we have five choices: $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$, $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/125\mathbb{Z}$, $\mathbb{Z}/625\mathbb{Z}$, $\mathbb{Z}/25\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z}$.

2. **Let $\mathbb{Z}(p^\infty)$ be the Prüfer $p$-group. Prove or disprove: $\mathbb{Z}(p^\infty) \approx \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)$.**

   **Disproof:** The first one has $p-1$ elements of order $p$, the second one has $p^2-1$ elements of order $p$, so that these two groups cannot be isomorphic.

3. **Show that a subgroup of index 2 of a group is necessarily normal.**

   **Proof:** Let $H$ be a subgroup of index 2 of $G$. Let $a \in G \setminus H$. Then $G = H \uplus Ha = H \uplus aH$, so that $aH = G \setminus H = Ha$, hence $aH = Ha$. If $a \in H$, $aH = Ha$ as well. So $aH = Ha$ all $a \in G$ and $H \triangleleft G$.

4. **Show that $\mathbb{Q}^* \approx (\mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_{\omega} \mathbb{Z})$.**

   **Proof:** Let $q \in \mathbb{Q}^*$. Then $q = a/b$ for some $a, b \in \mathbb{Z} \setminus \{0\}$. Decomposing $a$ and $b$ into their prime factorization, we can write $q$ as a product of (negative or positive) powers of prime numbers. Set,

   $$ q = \varepsilon(q) \prod_{p \text{ prime}} p^{\text{val}_p(q)} $$

   where $\text{val}_p(q) \in \mathbb{Z}$ and $\varepsilon(q) = \pm 1$ depending on the sign of $q$. Note that all the $\text{val}_p(q)$ are 0 except for a finite number of them. Let $\varphi : \mathbb{Q}^* \rightarrow (\mathbb{Z}/2\mathbb{Z}) \oplus (\oplus_{\omega} \mathbb{Z})$ be defined by

   $$ \varphi(q) = (\varepsilon(q), \text{val}_2(q), \text{val}_3(q), \text{val}_5(q), \ldots) $$

   It is clear that $\varphi$ is an isomorphism of groups. (Here we view $\mathbb{Z}/2\mathbb{Z}$ as the multiplicative group $\{1, -1\}$).

5. **Find $|\text{Aut}(\mathbb{Z}/p^n\mathbb{Z})|$.**

   **Solution.** The group $\mathbb{Z}/p^n\mathbb{Z}$ being cyclic (generated by $1$, the image of 1), any endomorphism $\varphi$ of $\mathbb{Z}/p^n\mathbb{Z}$ is determined by $\varphi(1)$. Then $\varphi(\pm x) = x\varphi(1)$ for all $x \in \mathbb{Z}$. Conversely any $a \in \mathbb{Z}/p^n\mathbb{Z}$ gives rise to a homomorphism $\varphi_a$ via $\varphi_a(\pm x) = x\varphi(1)$. In other words $\text{End}(\mathbb{Z}/p^n\mathbb{Z}) \approx \mathbb{Z}/p^n\mathbb{Z}$ via $\varphi \mapsto \varphi(1)$.
as rings with identity. Thus Aut(Z/p^nZ) = End(Z/p^nZ)^* \approx (Z/p^nZ)^* = \{a : a \text{ prime to } p\} = \{a : a \text{ not divisible by } p\} = Z/p^nZ \setminus pZ/p^nZ and has \(p^n - p^{n-1}\) elements.

6. What is Hom(Z/8Z, Z/6Z)? More generally, what is Hom(Z/nZ, Z/mZ)?

How many elements does it have? (15 pts.)

Answer: Since Z/nZ is cyclic and generated by 1 (the image of 1 in Z/nZ), any element \(\varphi\) of Hom(Z/nZ, Z/mZ) is determined \(\varphi(1) \in Z/mZ\). Let

\[\text{val}_1 : \text{Hom}(Z/nZ, Z/mZ) \to Z/mZ\]

be the map determined by \(\text{val}_1(\varphi) = \varphi(1)\). This is a homomorphism of (additive) groups. Furthermore it is one to one. However \(\text{val}_1\) is not onto as in Question 5, because not all \(a \in Z/mZ\) gives rise to a well-defined function \(x \mapsto x a\).

Claim: An element \(a \in Z/mZ\) gives rise to a well-defined function \(x \mapsto x a\) if and only if \(m/d\) divides \(a\) where \(d = \gcd(m, n)\).

Proof of the Claim: Assume \(m/d\) divides \(a\) where \(d = \gcd(m, n)\). We want to show that the map \(x \mapsto x a\) from Z/nZ into Z/mZ is well-defined. Indeed assume \(x = y\). Then \(n\) divides \(x - y\). So \(na\) divides \(xa - ya\). By hypothesis, it follows that \(nm/d\) divides \(xa - ya\). Since \(nm/d = \operatorname{lcm}(m, n)\), we get that \(\operatorname{lcm}(m, n)\) divides \(xa - ya\). Hence \(m\) divides \(xa - ya\). It follows that \(xa = ya\).

Conversely, assume that the function \(x \mapsto xa\) from Z/nZ into Z/mZ is well-defined. Then \(na = 0a = 0\) and \(m\) divides \(na\). Hence \(m/d\) divides \((n/d)a\). Since \(n/d\) and \(m/d\) are prime to each other we get that \(m/d\) divides \(a\). This proves the claim.

Now we continue with the solution of our problem. The claim shows that the homomorphism

\[\text{val}_1 : \text{Hom}(Z/nZ, Z/mZ) \to (m/d)Z/mZ\]

is an isomorphism. We can go further and prove that \((m/d)Z/mZ \approx Z/dZ\)

Claim: If \(n = mp\) then \(Z/nZ \approx Z/pZ\).

Proof of the Claim: Let \(\varphi : Z \to mZ/nZ\) be defined by \(\varphi(x) = mx\).

Clearly \(\varphi\) is a homomorphism and onto. Its kernel is \(\{x \in Z : n\) divides \(mx\}\) = \(\{x \in Z : mp\) divides \(mx\}\) = \(\{x \in Z : p\) divides \(x\}\) = \(pZ\). So \(Z/pZ \approx mZ/mZ\).

Thus \(\text{Hom}(Z/nZ, Z/mZ) \approx Z/dZ\) where \(d = \gcd(m, n)\).

7. Let \(p\) be a prime, \(A\) a finite \(p\)-group and \(\varphi \in \text{Aut}(A)\) an automorphism of order \(p^n\) for some \(n\). Show that \(\varphi(a) = a\) for some \(a \in A^\#\). (10 pts.)

Proof: Let \(G = \langle \varphi \rangle\). Then \(|G| = p^n\) and \(G\) acts on \(A^\#\). For \(a \in A^\#\), there is a bijection between the \(G\)-orbit \(Ga\) of \(a\) and the coset space \(G/Ga\) where \(Ga = \{g \in G : g(a) = a\}\) given by \(gGa \mapsto ga\). Thus \(|Ga| = |G/Ga|\) and \(|A^\#| = |A^\#| = |Ga| = |G/Ga|\).

If \(Ga \neq G\) for all \(a\), then \(|G/Ga| = p^i\) for some \(i \geq 1\) so that \(p\) divides \(|G/Ga| = |A^\#| = p^n - 1\), a contradiction. Thus \(Ga \neq G\) for some \(a\) and for this \(a\), \(|Ga| = 1\), i.e. \(Ga = \{a\}\) and \(\varphi(a) = a\).
8. Let \( G \) be a group and \( g \in G^\# \). Show that there is a subgroup \( H \) of \( G \) maximal with respect to the property that \( g \not\in H \). (10 pts.)

**Proof:** Let \( Z = \{ H \leq G : g \not\in H \} \). Order \( Z \) by inclusion. Since the trivial group \( 1 \in Z \), \( Z \neq \emptyset \). It is easy to show that if \( (H_t)_t \) is an increasing chain from \( Z \) then \( \bigcup_t H_t \in Z \). Thus \( Z \) is an inductive set. By Zorn’s Lemma it has a maximal element, say \( H \). Then \( H \) is a maximal subgroup of \( G \) not containing \( g \).

9. A group \( G \) is called divisible if for every \( g \in G \) and \( n \in \mathbb{N} \setminus \{0\} \) there is an \( h \in G \) such that \( h^n = g \).

9a. Show that a divisible group cannot have a proper subgroup of finite index. (10 pts.)

**Proof:** Assume \( G \) is divisible. Let \( H \triangleleft G \) be a subgroup of finite index, say \( n \). We first prove that \( G \) has a normal subgroup \( K \) of finite index contained in \( H \).

**Claim:** A group \( G \) that has a subgroup of index \( n \) has a normal subgroup of index dividing \( n! \) and contained in \( H \).

**Proof of the Claim.** Let \( G \) act on the left coset space \( G/H \) via \( g.(xH) = gxH \). This gives rise to a homomorphism \( \varphi \) from \( G \) into \( \text{Sym}(G/H) \), and the latter is isomorphic to \( \text{Sym}(n) \). Thus \( \text{Ker}(\varphi) \) is a normal subgroup and \( \varphi \) gives rise to an embedding of \( G/\text{Ker}(\varphi) \) into \( \text{Sym}(n) \). Thus \( |G/\text{Ker}(\varphi)| \) divides \( n! \) and \( \text{Ker}(\varphi) \) is a normal subgroup of index dividing \( n! \).

An easy calculation shows that \( \text{Ker}(\varphi) = \{ g \in G : g(xH) = xH \text{ all } g \in G \} = \bigcap_{x \in G} H^x \leq H \). This proves the claim.

Let \( K \) be the normal subgroup of index \( m \) of \( G \). Let \( a \in G \). Let \( b \in G \) be such that \( a = b^m \). Then \( a = b^m \in K \) (because the group \( G/K \) has order \( m \)) and so \( G = K \).

9b. Conclude that a divisible abelian group cannot have a proper subgroup which is maximal with respect to being proper. (10 pts.)

**Proof:** Let \( G \) be a divisible abelian group. Let \( H \triangleleft G \) be a maximal subgroup of \( G \). Then \( G/H \) has no nontrivial proper subgroups. Thus \( G/H \) is generated by any of its nontrivial elements. In particular \( G/H \) is cyclic. Since \( G/H \) cannot be isomorphic to \( \mathbb{Z} \) (because \( \mathbb{Z} \) has proper nontrivial subgroups, like \( 2\mathbb{Z} \)), \( G/H \) is finite. By the question above \( H = G \).

10. Let \( G \) be a group. Let \( H \triangleright G \).

10a. Assume \( Z \cong H \). Show that \( C_G(H) \) has index 1 or 2 in \( G \). (10 pts.)

**Proof:** Any element of \( G \) gives rise to an automorphism of \( H \) (hence of \( Z \)) by conjugation. In other words, there is a homomorphism of groups \( \varphi : G \to \text{Aut}(H) \cong \text{Aut}(Z) \) given by \( \varphi(g)(h) = h^g \) for all \( h \in G \). The kernel of \( \varphi \) is clearly \( C_G(H) \). Thus \( G/C_G(H) \) embeds in \( \text{Aut}(Z) \). But \( Z \) has only two generators, 1 and -1 and any automorphism of \( Z \) is determined by its impact on 1, which must be 1 or -1. Thus \( |\text{Aut}(Z)| = 2 \). This proves it.

10b. Assume \( H \) is finite. Show that \( C_G(H) \) has finite index in \( G \). (5 pts.)

**Proof:** As above. \( \varphi \) is a homomorphism from \( G \) into the finite group \( \text{Aut}(H) \) and the kernel of this automorphism is \( C_G(H) \).