Topology Homework
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Let X be a topological space and A a subset of X. An **open cover** of A is a collection $(U_i)_{i\in I}$ of open subsets U_i of X such that $A \subseteq \cup_{i\in I} U_i$. A **subcover** of an open cover $(U_i)_{i\in I}$ is an open cover of the form $(U_i)_{i\in J}$ for some subset J of I. The subcover $(U_i)_{i\in J}$ of the cover cover $(U_i)_{i\in I}$ is called finite if the index set J is finite.

A subset K of a topological space X is called compact if every open cover of K has a finite subcover.

1. Show that the union of finitely many compact subsets is compact.
2. Find compact subsets of a discrete space. (A topological space is **discrete** if every subset is open).
3. Show that a compact subset of a metric space is bounded.
4. Show that a compact subset of a metric space is closed.
5. Find an example of a metric space with a noncompact closed and bounded subset.
6. Let X be a metric space. Let $(x_n)_n$ be a sequence converging to $x \in X$. Show that the set $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is compact.
7. Show that the open interval $(0, 1)$ of $\mathbb{R}$ (with the Euclidean topology) is not compact.
8. Let $X = \mathbb{R} \cup \{\infty\}$ where ∞ is a new symbol. Consider the topology generated by all sets of the form (a, b) and $(-\infty, a) \cup (b, \infty) \cup \{\infty\}$ for $a, b \in \mathbb{R}$. Show that X is compact.
9. Is the topological space X above a metric space?
10. Show that a closed and bounded subset of $\mathbb{R}$ is compact.