1. Let $K$ be an algebraically closed field. Find the number of conjugacy classes of elements of $\text{GL}_5(K)$ whose characteristic polynomial is $(T - 2)^5$.

2. Let $\pi$ be a set of primes. A torsion group is called a $\pi$-group if the prime divisors of its elements’ orders are in $\pi$. Show that every group has a maximal $\pi$-subgroup.

3. Let $R$ be a ring and $M$ an $R$-module. Let $\varphi : M \to R$ be an $R$-module epimorphism. Show that $M \cong \ker(\varphi) \oplus R$. [Hint: Let $e \in M$ be such that $\varphi(e) = 1$. We have $m = (m - \varphi(m)e) + \varphi(m)e$.]

4. Let $R$ be a ring and $G$ a group. Consider the set $R[G]$ of the formal sums $\sum_{g \in G} r_g g$ where $r_g \in R$ for all $g$ and only finitely many of them is nonzero (in other words, by ignoring $0g$’s we may assume that the sum is finite). $R[G]$ becomes a ring and a left $R$-module with the following operations:

$$\left( \sum_{g \in G} r_g g \right) + \left( \sum_{g \in G} s_g g \right) = \sum_{g \in G} (r_g + s_g) g$$

$$\left( \sum_{g \in G} r_g g \right) \left( \sum_{g \in G} s_g g \right) = \sum_{g \in G, \, \eta_k s_k = g} \left( \sum_{h \in G} r_{hk} \right) g$$

$$r \left( \sum_{g \in G} r_g g \right) = \sum_{g \in G} (rr_g) g .$$

4a. If $R$ has $n$ elements and $G$ has $m$ elements, how many elements does $R[G]$ have?

4b. Let $R$ be the ring $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ and $G$ be the group $\mathbb{Z}/2\mathbb{Z} = \{1, x\}$ (with $x^2 = 1$). Write the elements of $R[G]$. Find its invertible and nilpotent elements.

4c. Show that $R^* \leq R[G]^*$ and $G \leq R[G]^*$.

Define the map $\varphi : R[G] \to R$ by the rule

$$\varphi(\sum_{g \in G} r_g g) = \sum_{g \in G} r_g .$$

4d. Show that $\varphi$ is a ring and an $R$-module epimorphism.

4e. Apply Question 2 to $R[G]$ and $\varphi$.

4f. Show that, for $g \in G$, $1 - g \notin R[G]^*$.

4g. Show that for $g \in G$, $1 + g \notin R[G]^*$.

4h. Let $p \neq 2$ be a prime. Show that if $R$ is a ring of characteristic $p$ and $g \in G$ has order $p$, then $1 + g \notin R[G]^*$.

4i. Let $R$ be a domain and $G$ be the group $\mathbb{Z} = \langle x \rangle$. Show that $R[G]^* = \{rx^n : r \in R^*, n \in \mathbb{Z}\}$. 