Topology

First Midterm
Gümüşlük Akademisi
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Throughout, X stands for a topological space.

1. Let $U \subseteq X$ be open and $C \subseteq X$ be closed. Show that $U \setminus C$ is open in X. (2 pts.)

2. Show that X is Hausdorff iff the diagonal $\{(x, x) : x \in X\}$ is closed in the product topology of $X \times X$. (5 pts.)

3. For $A \subseteq X$, define the **exterior** of A as $\text{ext}(A) = \text{int}(A^c)$.

3a. Show that $\text{ext}(A) = \underline{A}^c$ for all $A \subseteq X$.

3b. Show that, for two subsets A and B of X,

$$\text{ext}(A \cup B) = \text{ext}(A) \cap \text{ext}(B) \subseteq \text{ext}(A) \cup \text{ext}(B) \subseteq \text{ext}(A \cap B).$$

(8 pts.)

3c. Show that $\underline{A} \setminus \text{int}(A) = (\text{int}(A) \cup \text{ext}(A))^c$. (8 pts.)

4. Let (X, d) be a metric space.

4a. Show that $|d(x, y) - d(a, b)| \leq d(x, a) + d(y, b)$ for all $a, b, x, y \in X$. (4 pts.)

4b. Show that the distance map $d : X \times X \to \mathbb{R}$ is continuous. (**Hint:** On $X \times X$ choose the taxi-cab metric and apply part a). (8 pts.)

5. Let X and Y be metric spaces, $f : X \to Y$ a function and $x \in X$. Show that f is continuous at a iff for every sequence $(x_n)_n$ of X that converges to a, $\lim_{n\to\infty} f(x_n) = f(a)$. (8 pts.)

6. Let X and Y be metric spaces. Let $(x_n)_n$ and $(y_n)_n$ be sequences of X and Y respectively. Show that $\lim_{n\to\infty} (x_n, y_n) = (x, y)$ iff $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. (8 pts.)

7. Let ∞ and $-\infty$ be two new symbols. Set $\mathbb{R}^* = \mathbb{R} \cup \{\infty, -\infty\}$. Order $\mathbb{R}^*$ as one should. Consider the topology on X^* generated by the open intervals of $\mathbb{R}$ together with sets of the form $[-\infty, r)$ and $(r, \infty]$ for $r \in \mathbb{R}$. Show that $\mathbb{R}^*$ is compact for that topology. (8 pts.)

8. Show that X is compact iff for every family $(C_i)_i$ of closed subsets, $\cap_i C_i \neq \emptyset$ whenever every finite intersection of the sets C_i is nonempty. (8pts.)

9. For $n > 0$, let $f : \mathbb{R} \to \mathbb{R}$ be given by $f_n(x) = \dfrac{\sin(nx)}{n}$. Show that the functions f_n converge to 0 uniformly, but that the derivatives f_n' do not even converge pointwise to any function. (10 pts.)

10. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{1}{1+x^2}$ is uniformly continuous. (10 pts.)

11. Find a metric space with no countable dense subset. (13 pts.)