Topology HW14

Uniform Continuity

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Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a continuous function. Recall that this means that

$$\forall a \in X, \forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall x \in X,\ d_Y(f(x), f(a)) < \varepsilon \text{ whenever } d_X(x, a) < \delta.$$

Here δ depends on ε and on a, that is why we sometimes write $\delta_{\varepsilon, a}$ instead of just δ. The function f is called **uniformly continuous** if δ can be chosen independently from a; this means that

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ such that } \forall a, x \in X,\ d_Y(f(x), f(a)) < \varepsilon \text{ whenever } d_X(x, a) < \delta.$$

1. Show that any constant function is uniformly continuous.

2. Show that if $X = Y$, the identity function is uniformly continuous.

3. Let $X = Y = \mathbb{R}$. Let $a, b \in \mathbb{R}$. Show that the function f given by $f(x) = ax + b$ is uniformly continuous.

4. Let $X = Y = \mathbb{R}$. Show that the function f given by $f(x) = x^2$ is not uniformly continuous.

5. Show that the composition of two uniformly continuous functions is uniformly continuous.

6. Find compact metric spaces X and Y and a continuous function from X into a Y that is not uniformly continuous.

7. Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{1}{1+x^2}$ is uniformly continuous.