Let $X$ be a set and $(Y, d)$ a metric space. Let $(f_n)_n$ be a sequence of functions from $X$ into $Y$. Given an element $x \in X$, we can form the sequence $(f_n(x))_n$ of $Y$. This sequence may or may not converge in $Y$. Assume it converges to some element of $Y$. Let us call this element, that depends on $X$, $f(x)$. Thus $f(x)$ is defined to be $\lim_{n \to \infty} f_n(x)$ in case this limit exists. As we know this means that 
\[ \forall \varepsilon > 0 \exists N \text{ such that for } n > N, d(f_n(x), f(x)) < \varepsilon. \]

Note that the natural number $N$ depends on $\varepsilon$ but also on $x$; for this reason we sometimes write $N = N_{\varepsilon, x}$.

In case the sequence $(f_n(x))_n$ converges for all $x \in X$, we can define the function $f$ from $X$ into $Y$ by the rule $\lim_{n \to \infty} f_n(x)$. We then say that the sequence $(f_n)_n$ of functions converges pointwise to the function $f$, in this case we write $\lim_{n \to \infty} f_n = f$ iff 
\[ \forall x \in X \forall \varepsilon > 0 \exists N \text{ such that for } n > N, d(f_n(x), f(x)) < \varepsilon. \]

1. Let $Y = \mathbb{R}$ and, for $n > 0$, let $f_n : X \to Y$ be the constant function $1/n$. Show that $\lim f_n$ exists. Show that the natural number $N$ of the definition can be chosen to depend only on $\varepsilon$.

2. Let $X = Y = \mathbb{R}$ and let $f_n : X \to Y$ be the function $x/n$. Show that $\lim f_n$ exists. Show that the natural number $N$ of the definition cannot be chosen to be independent of $x$.

3. Let $X = [-1, 2]$ $Y = [0, 1]$ and for $n > 0$, let $f_n : X \to Y$ be the function defined by
   \[
   f_n(x) = \begin{cases} 
   1 & \text{if } -1 \leq x \leq 0 \text{ or } 1 \leq x \leq 2 \\
   -nx + 1 & \text{if } 0 \leq x \leq 1/n \\
   0 & \text{if } 1/n \leq x \leq 1 - 1/n \\
   nx + 1 - n & \text{if } 1 - 1/n \leq x \leq 1.
   \end{cases}
   \]
   3a. By sketching the graph of $f_n$, check that $f_n$ is continuous for all $n = 1, 2, 3, \ldots$.
   3b. Show that the sequence $(f_n)_n$ of functions converges pointwise to a noncontinuous function.

4a. Find a sequence $f_n : [0, 1] \to [0, 1]$ of continuous functions that converges pointwise to a function $f : [0, 1] \to [0, 1]$ which is nowhere continuous.

4b. A function $f : [0, 1] \to [0, 1]$ is called piecewise linear if its graph consists of a union of finitely many line segment. The length $l(f)$ of a piecewise linear function is the sum of the lengths of the line segments. Find a sequence $(f_n)_n$ of continuous piecewise linear functions that converges pointwise to a continuous piecewise linear function $f$ such that $\lim_{n \to \infty} l(f_n) \neq l(f)$.

(Sınav tamamlanamamıştır.)