Topology HW12

Pointwise and Uniform Convergence

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Let X be a set and (Y, d) a metric space. Let $(f_n)_n$ be a sequence of functions from X into Y. Given an element $x \in X$, we can form the sequence $(f_n(x))_n$ of Y. This sequence may or may not converge in Y. Assume it converges to some element of Y. Let us call this element, that depends on X, $f(x)$. Thus $f(x)$ is defined to be $\lim_{n\to\infty} f_n(x)$ in case this limit exists. As we know this means that

$$\forall \varepsilon > 0\ \exists N \text{ such that for } n > N,\ d(f_n(x), f(x)) < \varepsilon.$$

Note that the natural number N depends on ε but also on x; for this reason we sometimes write $N = N_{\varepsilon, x}$.

In case the sequence $(f_n(x))_n$ converges for all $x \in X$, we can define the function f from X into Y by the rule

$$f(x) = \lim_{n\to\infty} f_n(x).$$

We then say that the sequence $(f_n)_n$ of functions **converges pointwise** to the function f, in this case we write $\lim_{n\to\infty} f_n = f$. Thus $\lim_{n\to\infty} f_n = f$ iff

$$\forall x \in X\ \forall \varepsilon > 0\ \exists N \text{ such that for } n > N,\ d(f_n(x), f(x)) < \varepsilon.$$

1. Let $Y = \mathbb{R}$ and, for $n > 0$, let $f_n : X \to Y$ be the constant function $1/n$. Show that $\lim f_n$ exists. Show that the natural number N of the definition can be chosen to depend only on ε.

2. Let $X = Y = \mathbb{R}$ and let $f_n : X \to Y$ be the function x/n. Show that $\lim f_n$ exists. Show that the natural number N of the definition cannot be chosen to be independent of x.

3. Let $X = [-1, 2]$ $Y = [0, 1]$ and for $n > 0$, let $f_n : X \to Y$ be the function defined by

$$\begin{aligned} f_n(x) &= 1 && \text{if } -1 \le x \le 0 \text{ or } 1 \le x \le 2 \\ f_n(x) &= -nx + 1 && \text{if } 0 \le x \le 1/n \\ f_n(x) &= 0 && \text{if } 1/n \le x \le 1 - 1/n \\ f_n(x) &= nx + 1 - n && \text{if } 1 - 1/n \le x \le 1. \end{aligned}$$

3a. By sketching the graph of f_n, check that f_n is continuous for all $n = 1, 2, 3, \ldots$

3b. Show that the sequence $(f_n)_n$ of functions converges pointwise to a noncontinuous function.

4a. Find a sequence $f_n : [0, 1] \to [0, 1]$ of continuous functions that converges pointwise to a function $f : [0, 1] \to [0, 1]$ which is nowhere continuous.

4b. A function $f : [0, 1] \to [0, 1]$ is called **piecewise linear** if its graph consists of a union of finitely many line segment. The **length** $l(f)$ of a piecewise linear function is the sum of the lengths of the line segments. Find a sequence $(f_n)_n$ of continuous piecewise linear functions that converges pointwise to a continuous piecewise linear function f such that $\lim_{n\to\infty} l(f_n) \neq l(f)$.

(Sınav tamamlanamamıştır.)