Topology HW12

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Throughout X stands for a topological space.

1. For $A \subseteq X$, set $\alpha(A) = (\underline{A})^{\circ}$ and $\beta(A) = \underline{A^{\circ}}$. Thus $\alpha(A)$ is the interior of the closure of A and $\beta(A)$ is the closure of the interior of A. Recall that $\delta A = \underline{A} \cap \underline{A^c}$.

1a. Show that $\alpha(\alpha(A)) = \alpha(A)$ and $\beta(\beta(A)) = \beta(A)$ for all $A \subseteq X$.

1b. Show that if A and B are disjoint and open, then $\alpha(A)$ and $\alpha(B)$ are also disjoint.

1c. Find a subset A of $\mathbb{R}$ such that the subsets A, A°, $\underline{A}$, $\alpha(A)$, $\beta(A)$, $\beta(\underline{A})$, $\alpha(A^{\circ})$ are all distinct.

1d. For $A, B \subseteq X$, show that $\delta(\underline{A}) \subseteq \delta(A)$ and $\delta(A^{\circ}) \subseteq \delta(A)$.

1e. For $A, B \subseteq X$, show that $\delta(A \cup B) \subseteq \delta(A) \cup \delta(B)$. Does the equality always hold? Show that if $\underline{A} \cap \underline{B} = \emptyset$, then $\delta(A \cup B) = \delta(A) \cup \delta(B)$.

1f. Let Y be another topological space and let $A \subseteq X$ and $B \subseteq Y$. Show that $\delta(A \times B) = (\delta(A) \times \underline{B}) \cup (\underline{A} \times \delta(B))$.

2. Recall that if $Y \subseteq X$, we can endow Y with a topological space structure by declaring that the open subsets of Y are the intersection with Y of open subsets of X.

Let $B \subseteq A \subseteq X$. The subsets A and B have the induced topology from the topology of X, call τ_A and τ_B these two topologies. But B can also have the induced topology from τ_A. Show that these two topologies on B are the same.

3. Let $K \subseteq X$ be compact and $C \subseteq X$ be closed. Suppose that $C \subseteq K$. Show that C is compact in X.

4. Suppose X is Hausdorff.

4a. Let A and B be two compact and disjoint subsets of X. Show that there are disjoint open subsets U and V of X such that $U \cap V = \emptyset$.

4b. Show that every compact subset of X is closed.

4c. Let Y be a compact space and let $f : Y \to X$ be continuous. Show that the image of a closed subset of Y under f is closed.

4d. Let Y be a compact space and let $f : Y \to X$ be a continuous bijection. Show that f is a homeomorphism (i.e. f^{-1} is also continuous).

5. Assume X is compact and let Y be a topological space. Let $y \in Y$ and W be an open subset of $X \times Y$ such that $X \times \{y\} \subseteq W$. Show that there is an open subset $V \subseteq Y$ such that $X \times \{y\} \subseteq X \times V \subseteq W$.

6. Show that the product of two compact spaces is compact. (Hint: Use #5).