Throughout $X$ stands for a topological space.

1. For $A \subseteq X$, set $\alpha(A) = (A)^o$ and $\beta(A) = A^\circ$. Thus $\alpha(A)$ is the interior of the closure of $A$ and $\beta(A)$ is the closure of the interior of $A$. Recall that $\delta A = A \cap A^c$.

1a. Show that $\alpha(\alpha(A)) = \alpha(A)$ and $\beta(\beta(A)) = \beta(A)$ for all $A \subseteq X$.

1b. Show that if $A$ and $B$ are disjoint and open, then $\alpha(A)$ and $\alpha(B)$ are also disjoint.

1c. Find a subset $A$ of $\mathbb{R}$ such that the subsets $A, A^o, \alpha(A), \beta(A), \alpha(A^o)$ are all distinct.

1d. For $A, B \subseteq X$, show that $\delta(A) \subseteq \delta(A)$ and $\delta(A^o) \subseteq \delta(A)$.

1e. For $A, B \subseteq X$, show that $\delta(A \cup B) \subseteq \delta(A) \cup \delta(B)$. Does the equality always hold? Show that if $A \cap B = \emptyset$, then $\delta(A \cup B) = \delta(A) \cup \delta(B)$.

1f. Let $Y$ be another topological space and let $A \subseteq X$ and $B \subseteq Y$. Show that $\delta(A \times B) = (\delta(A) \times B) \cup (A \times \delta(B))$.

2. Recall that if $Y \subseteq X$, we can endow $Y$ with a topological space structure by declaring that the open subsets of $Y$ are the intersection with $Y$ of open subsets of $X$.

Let $B \subseteq A \subseteq X$. The subsets $A$ and $B$ have the induced topology from the topology of $X$, call $\tau_A$ and $\tau_B$ these two topologies. But $B$ can also have the induced topology from $\tau_A$. Show that these two topologies on $B$ are the same.

3. Let $K \subseteq X$ be compact and $C \subseteq X$ be closed. Suppose that $C \subseteq K$. Show that $C$ is compact in $X$.

4. Suppose $X$ is Hausdorff.

4a. Let $A$ and $B$ be two compact and disjoint subsets of $X$. Show that there are disjoint open subsets $U$ and $V$ of $X$ such that $U \cap V = \emptyset$.

4b. Show that every compact subset of $X$ is closed.

4c. Let $Y$ be a compact space and let $f : Y \to X$ be continuous. Show that the image of a closed subset of $Y$ under $f$ is closed.

4d. Let $Y$ be a compact space and let $f : Y \to X$ be a continuous bijection. Show that $f$ is a homeomorphism (i.e. $f^{-1}$ is also continuous).

5. Assume $X$ is compact and let $Y$ be a topological space. Let $y \in Y$ and $W$ be an open subset of $X \times Y$ such that $X \times \{y\} \subseteq W$. Show that there is an open subset $V \subseteq Y$ such that $X \times \{y\} \subseteq X \times V \subseteq W$.

6. Show that the product of two compact spaces is compact. (Hint: Use #5).