Topology HW11,5

(Connectedness)

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The topological space X is said to be **connected** if X is not the union of two disjoint nonempty open subsets. Note that X is connected if only if Ø and X are the only clopen subsets of X. A subset A of X is called **connected** if A is a connected space with its induced topology. Clearly, a singleton subset of X is always connected.

1. Is $\mathbb{Q}$ connected?

2. Let A be a connected subset of X. Show that if $A \subseteq B \subseteq \underline{A}$, then B is connected.

3. Show that $\mathbb{R}$ is connected.

4. Let $(A_i)_i$ be a family of connected subsets of X. Assume that for $i \neq j$, $A_i \cap A_j \neq \emptyset$. Show that $\cup_i A_i$ is connected.

5. Let $(A_i)_{i = 0, 1, 2, \ldots}$ be a sequence of connected subsets of X. Assume that for all i, $A_i \cap A_{i+1} \neq \emptyset$. Show that $\cup_i A_i$ is connected.

6. Let $A \subseteq X$ be connected. Let $B \subseteq X$. Assume that $A \cap B \neq \emptyset$ and $A \cap B^c \neq \emptyset$. Show that $A \cap \delta B \neq \emptyset$.

7. Deduce from above that if X is connected and $\emptyset \neq A \subset X$, then $\delta A \neq \emptyset$.

8. Let $x \in X$. Show that the union of connected subsets of X that contain x is the largest connected subset of X containing x. This subset is called **connected component** of x.

9. Show that connected components of points of x partition X.

10. Let $X = \mathbb{Q}$. What is connected component of any $q \in \mathbb{Q}$?

11. Find an example where the connected components are not closed.

12. Show that the connected components of a space are closed.

13. Show that if the connected components are open, then they are also closed.

14. Show that if X has finitely many connected components, then each connected component is clopen.

15. Show that the image of a connected set under a continuous map is connected.

16. Show that the product of two connected spaces is connected.