Topology HW10

Gümüşlük Akademisi
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Ali Nesin

The symbol X will always stand for a topological space.

1. If $U \subseteq X$ is open, is it true that $U = \underline{U}^{o}$? Justify your answer.

2. Let $Y \subseteq X$ be a subset. Consider Y as a topological space (with the induced topology[1]). Let $A \subseteq Y$.

2a. Take $X = \mathbb{R}$ with the usual topology. Let $Y = [0, 1)$. Find a countable base for the induced topology on Y.

2b. Show that if A is closed (resp. open) in Y and Y is closed (resp. open) in X then A is closed (resp. open) in X.

2c. Show that the closure of A in Y is equal to $\underline{A} \cap Y$ where $\underline{A}$ stands for the closure of A in X.

3. Let X and Y be topological spaces. Let $A \subseteq X$ and $B \subseteq Y$. Show that

$$\underline{A \times B} = \underline{A} \times \underline{B}.$$

4. Let $A \subseteq X$. Show that $x \in \underline{A}$ iff every open subset of X containing x intersects A nontrivially.

5a. Let $X = \mathbb{R}$. Find the set of limit points[2] of the following subsets of $\mathbb{R}$.

i) $A = [0, 1]$
ii) $B = (0, 1]$
iii) $C = (0, 1)$
iv) $D = \mathbb{Z}$
v) $E = \{1/n : n = 1, 2, 3, ...\}$
vi) $F = \mathbb{Q}$
vii) $G = \{a/2^n : a \in \mathbb{Z}, n \in \mathbb{N}\}$
viii) $H = \{(x, y) \in \mathbb{R}^2 : x = y\}$

5b. Let $A \subseteq X$. Show that a point of X is a limit point of A iff it belongs to the closure of $A \setminus \{x\}$.

5c. Let $A \subseteq X$. Let A' be the set of limit points of A. Show that $\underline{A} = A \cup A'$.

5d. Show that a subset of a topological space is closed iff it contains all its limit points.

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[1] Declare a subset V of Y "open" if it is the intersection of an open subset of X with Y. These open subsets of Y induce a topology on Y called the **induced topology**. Note for the next questions that, for a subset of Y, to be closed or open in Y may be different from being open or closed in X.

[2] Let $A \subseteq X$. An element $x \in X$ is called a **limit point** or a **cluster point** or a **point of accumulation** of A (in X) if every open subset of X containing x intersects A in a point of A different from x. Note that x may or may not be in A.

6. Assume that X satisfies[3] T_1.

6a. Show that every finite set of points of X is closed.

6b. Let $A \subseteq X$. Show that a point $x \in X$ is a limit point of x iff every open subset of X containing x contains infinitely many points of A.

7a. Let $X = \mathbb{R}$ endowed with the usual topology. Find the boundary[4] of the subsets in # 4a of $\mathbb{R}$.

7b. Let $A \subseteq X$. Show that if $A^{o} \cap \delta A = \emptyset$, then $\underline{A} = A^{o} \cup \delta A$.

7c. Show that $A \subseteq X$ is open iff $\delta A = \underline{A} \setminus A$.

8. Let $f : X \to Y$ be a map between two topological spaces. Show that f is continuous iff for every subset $A \subseteq X$, $f(\underline{A}) \subseteq \underline{f(A)}$.

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[3] X is said to satisfy T_1 if given any two distinct points a, b of X there is an open set that contains a but not b.

[4] For a subset A of a topological space X, the **boundary** δA of A in X is defined as $\underline{A} \cap \underline{(X \setminus A)}$.