Topology HW9

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X and Y stand for topological spaces.

A subset $C \subseteq X$ is called **compact** if any open covering of C has a finite subcovering.

1. Show that a function $f : X \to Y$ is continuous iff the inverse image of closed subsets of Y are closed in X.

2a. Let $\pi_1 : X \times Y \to X$ be the first projection. Show that the image of an open subset of $X \times Y$ under π_1 is open in X.

2b. Show that the image of a closed subset of $X \times Y$ under π_1 is not necessarily closed in X.

2c. Find topological spaces X and Y and a continuous function $f : X \to Y$ such that the image of open subsets of X are not necessarily open in Y.

3a. Show that $\mathbb{R}$ is not a compact subset of $\mathbb{R}$ under the usual Euclidean topology.

3b. Show that the image of a compact subset under a continuous map is compact.

3c. Is it true that the inverse image of a compact subset under a continuous map is always compact?

4a. Let $(a_i)_i$ and $(a_i)_i$ be an increasing and decreasing sequence of $\mathbb{R}$. Show that the intersection $\bigcap_i [a_i, b_i]$ is nonempty.

4b. Show that a closed bounded interval $[a, b]$ of $\mathbb{R}$ is compact. (**Hint:** Assume not. Consider an open covering of $[a, b]$ without a finite subcovering. One of $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ does not have a finite subcovering. Repeat the argument with this closed subinterval, to get a chain of closed intervals of length half of the preceding one. Apply part 4a.)

4c. Is it true that closed bounded subsets of $\mathbb{R}$ are compact?

4d. Show that a compact subset C of a metric space is bounded, i.e. there is an r such that for all $x, y \in C$, $d(x, y) < r$.

4e. Show that a compact subset C of a metric space is closed. (**Hint:** Let $x \in C^c$. For every $c \in C$ there are disjoint open subsets U_c containing c and V_c containing x (why). The open subsets $(U_c)_{c\in C}$ cover C).