Topology HW8

(Continuous Functions)

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Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $a \in X$. A map $f : X \to Y$ is called **continuous at** a if for any $\varepsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(x), f(a)) < \varepsilon$ whenever $d_X(x, a) < \delta$. The function f is called **continuous** if it is continuous at every $a \in X$.

Let X and Y be two topological spaces. Let $a \in X$. A map $f : X \to Y$ is called **continuous at** a if for any open subset V of Y containing $f(a)$, the inverse image $f^{-1}(V)$ is an open subset of X. The function f is called **continuous** if it is continuous at every $a \in X$.

1. Let (X, d_X) and (Y, d_Y) be two metric spaces. Let $a \in X$. Show that a map $f : X \to Y$ is continuous with respect to the first definition iff it is continuous with respect to the second definition.

From now on we deal only with topological spaces and we adopt the second definition. From now on X, Y and Z stand for topological spaces.

2. Let X have discrete topology and Y any topological space. Find all continuous functions from X into Y.

3. Let X have rudest topology (its open subsets are $\varnothing$ and X) and Y any topological space. Find all continuous functions from X into Y.

4. Let T be any set and $f : T \to X$ any function. Show that there is a anique rudest topology on T that makes f continuous.

5. Show that a constant function from X into Y is continuous.

6. Show that the identity function Id_X is continuous.

7. Show that the composition of two continuous functions is continuous.

8. Let $\wp$ a base of Y and $f : X \to Y$ a map. Show that f is continuous iff $f^{-1}(B)$ is open for every $B \in \wp$.

9. Show that the function $+ : \mathbb{R}^2 \to \mathbb{R}$ defined by $+(x, y) = x + y$ is continuous.

10. Show that the function $\times : \mathbb{R}^2 \to \mathbb{R}$ defined by $\times(x, y) = xy$ is continuous.

11. Let $f : X \to Y$ and $g : X \to Z$ be continuous maps. Define $f \times g : X \to Y \times Z$ by the rule $(f \times g)(x) = (f(x), g(x))$. Show that $f \times g$ is continuous. (Here $Y \times Z$ has the product topology).

12. (Polynomial Function). Show that a polynomial function f from $\mathbb{R}$ into $\mathbb{R}$ is continuous. (A polynomial function is a function that sends x to $a_o + a_1x + ... + a_nx^n$ for some fixed $a_o, a_1, ..., a_n \in \mathbb{R}$.)

13. (Inclusion Map). Let $A \subseteq X$. Show that the inclusion map $i : A \to X$ (that send $a \in A$ into a) is continuous.

14. (Projection). Let $\pi_1 : X \times Y \to X$ be the first projection, i.e. $\pi(x, y) = x$ for all $x \in X$ and $y \in Y$. Show that π_1 is continuous. (Here $X \times Y$ has the product topology).

15. (Product Topology). Show that the topology on $X \times Y$ is the smallest topology that makes the projection maps π_1 and π_2 continuous.