Topology HW7

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1. Let Y be a topological space and $f : X \to Y$ be any map. Show that the set

$$\{f^{-1}(V) : V \text{ open in } Y\}$$

defines a topology on X.

2. Let (X, d) be a metric space, $(x_n)_n$ a sequence from X and $x \in X$ not equal to any of the x_n. Assume that $\inf\{d(x_n, x) : n \in \mathbb{N}\} = 0$ and that $(x_n)_n$ is Cauchy. Show that $\lim x_n = x$.

3. Let (X, d) be a metric space. Let $A \subseteq X$ be a subset of X. Show that A is closed iff any sequence of A that has a limit in X has a limit in A.

4. Let (X, d) be a metric space. Let $\Im$ be a set of a disjoint closed subsets of X. For $A, B \in \Im$, define $d'(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$.

4a. Show that $d'(A, B) = 0$ does not necessarily imply $A = B$.

4b. Show that the triangular inequality may also not hold.

5. Let (Y, d) be a metric space. Let $f : X \to Y$ be a one-to-one map. For $x_1, x_2 \in X$, define $d_f(x_1, x_2) = d(f(x_1), f(x_2))$.

5a. Show that d_f is a metric on X.

5b. Show that for $a \in X$ and $r \in \mathbb{R}^{>0}$, $B_X(a, r) = f^{-1}(B_Y(f(a), r))$.

5c. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x$ if $x \neq 0, 1$ and $f(x) = 1 - x$ if $x = 0, 1$. For x, y define $d(x, y) = |f(x) - f(y)|$. Find $B(0, 1/2)$, $B(1, 3)$, $B(0, 1)$.

5d. Let $a \in \mathbb{R}^{>0}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = ax$. For x, y define $d(x, y) = |f(x) - f(y)|$. Find $B(0, 1/2)$, $B(1, 1)$, $B(0, 1)$.

5e. Let $a \in \mathbb{R}$. Let $f(x) = x$ if $x \geq a$ and $f(x) = x - 1$ if $x < a$. For x, y define $d(x, y) = |f(x) - f(y)|$. Find $B(a, 1/2)$, $B(a, 1)$, $B(a + 1, 1/2)$, $B(a + 1, 1)$, $B(a + 1, 2)$, $B(a - 1, 1/2)$, $B(a - 1, 1)$, $B(a - 1, 2)$.

6. Let X be a topological space. Let $A \subseteq X$ be a subset. A set $\wp$ of subsets of X is said to **cover** A if $A \subseteq \cup \wp$. If the elements of $\wp$ are also open subsets of X then we say that $\wp$ is an **open covering** of A. The subset A is called **compact** if every open covering of A has a finite subset that covers A. The topological space X is called **compact** if X is a compact subset of X.

6a. Show that a finite subset of X is always open.

6b. Show that the union of finitely many compact subsets of X is compact.

6c. Assume X is a metric space. Let $(x_n)_n$ be a sequence of X that converges to some $x \in X$. Show that $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact subset of X.

6d. Show that the one-point compactification of any topological space is compact (see HW #6).

6e. Suppose A is a compact subset of X. Show that any closed subset (closed in the topology of X) of A is compact.

6f. Assume X is a metric space. Show that a compact subset A of X is bounded, i.e. the set $\{d(x, y) : x, y \in A\}$ is bounded.

6g. Assume X is a metric space. Show that a compact subset A of X is closed. **Hint:** Otherwise there is an $x \notin A$ such that every open subset that contains x intersects A. For $a \in A$, choose open sets U_a and V_a such that $a \in U_a$, $x \in V_a$ and $U_a \cap V_a = \emptyset$.