Topology HW6

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1. Let (X, d) be a metric space. Let $a \in X$ and $r \in \mathbb{R}^{>0}$.

1a. Show that for $B(a, r) \subseteq X \subseteq \underline{B}(a, r)$, we have $\underline{X} = \underline{B}(a, r)$. Here, B and $\underline{B}$ stand for the open and closed balls respectively.

1b. Does $\underline{X} = \underline{B}(a, r)$ imply $B(a, r) \subseteq X \subseteq \underline{B}(a, r)$?

2. (Product Topology). Let X and Y be topological spaces. The topology generated by sets of the form $U \times V$ where U and V are open subsets of X and Y respectively is called the product topology on $X \times Y$.

2a. If the only open subsets of X (resp. of Y) are $\emptyset$ and X (resp. Y), what are the open subsets of $X \times Y$?

2b. Show that if the topologies on X and Y are generated by the metrics d_X and d_Y, then the topology on $X \times Y$ is generated by the usual metric

$$d((x_1, y_1), (x_2, y_2) = \sqrt{d_X(x_1, x_2)^2 + d_X(y_1, y_2)^2}.$$

4. Let X be a topological space. Recall that a set $\wp$ of open subsets of X is called a **base** if every open subset of X is a union of sets from $\wp$.

4a. Show that a set $\wp$ of open subsets of X is a base if,

i) $\wp$ covers X, i.e. $\cup\wp = X$.

ii) For every $x \in X$ and open subset U containing x, there is $V \in \wp$ such that $x \in V \subseteq U$.

4c. Show that a set $\wp$ of subsets of X is a base of the topology it generates if for every U and $V \in \wp$ and $x \in U \cap V$, there is a $W \in \wp$ such that $x \in W \subseteq U \cap V$.

4d. Let X and Y be topological spaces. Let $\wp_X$ and $\wp_Y$ be bases of X and Y respectively. Show that the set

$$\wp = \{U \times V : U \in \wp_X \text{ and } V \in \wp_Y\}$$

is a base of the product topological space $X \times Y$.

5. (**One-point Compactification).** Let (X, d) be a metric space. Let ∞ be an element not in X. Let $X^* = X \cup \{\infty\}$, $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. For $x, y \in X^*$, define

$$d_1 : X^* \times X^* \to \mathbb{R} \cup \{\infty\}$$

by extending d and defining (for $x \in \mathbb{R}$)

$$d_1(x, \infty) = \infty$$
$$d_1(\infty, \infty) = 0.$$

Extend also the order of $\mathbb{R}$ to $\mathbb{R}^*$ by defining $r < \infty$ for all $r \in \mathbb{R}$.

Consider the subsets of X^* of the form

$$\{x \in X^* : d_1(a, x) < r\}$$

or

$$\{x \in X^* : d_1(a, x) > r\}$$

for some $a \in X^*$ and $r \in \mathbb{R}^*$. Call these sets **extended open balls**. The topology generated on X^* by the extended open balls is called **one-point compactification** of X.

5a. Show that X is a subspace of X^*.
5b. Show that the extended open balls form a basis of X^*.