Topology HW5

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Throughout X stands for a topological space.

1. Let A be a subset of X. Let

$$A^{o} = \cup \{U \subseteq X : U \text{ is open and } U \subseteq A\}$$
$$\underline{A} = \cap \{F \subseteq X : F \text{ is closed and } U \subseteq F\}$$

A^{o} and $\underline{A}$ are called the **interior** and the **closure** of A respectively.

1a. Show that $\underline{A}$ is closed, $A \subseteq \underline{A}$ and $\underline{A}$ is the smallest such subset of X.
1b. Show that A^{o} is open, $A^{o} \subseteq A$ and A^{o} is the largest such subset of X.
1c. Show that if $A \subseteq B$ then $A^{o} \subseteq B^{o}$.
1d. Show that if $A \subseteq B$ then $\underline{A} \subseteq \underline{B}$.
1e. Show that $A = \underline{A}$ iff A is closed.
1f. Show that $A = A^{o}$ iff A is open.
1g. Show that $\underline{A} = \underline{\underline{A}}$ and $A^{o} = A^{oo}$.
1h. Show that $(A \cap B)^{o} = A^{o} \cap B^{o}$.
1i. S how that $A^{o} \cup B^{o} \subseteq (A \cup B)^{o}$. Does the equality always hold?
1j. Show that $\underline{(A \cup B)} = \underline{A} \cup \underline{B}$.
1k. Show that $\underline{A \cap B} \subseteq \underline{A} \cap \underline{B}$. Does the equality always hold?

2. A subset A of X is called **dense** in X if $\underline{A} = X$.

2a. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$ (the usual topology).
2b. Suppose that the only open subsets of X are $\varnothing$ and X. What are the dense subsets of X?
2c. Suppose that no proper subset of X is dense in X. What can you say about the topology of X?

3. (Induced topology). Let $Y \subseteq X$. We call a subset of Y **open** if it is the intersection of Y with an open subset of X.
3a. Show that this defines a topology on Y.
We say that the topology on Y is **induced** from X and that Y is a subspace of X.

3b. Consider $\mathbb{Z}$ as a topological space induced from that of $\mathbb{R}$. What are the open subsets of $\mathbb{Z}$?
3c. Let $Y \subseteq X$ and consider Y as an induced topological space. A subset A of Y has a closure and an interior in X and in Y. Denote these by A^{oY}, A^{oX}, $\underline{A}_Y$, $\underline{A}_X$, the notation being self explanatory.
Let now $A \subseteq X$. What are the relation ships between

i) $\underline{A \cap Y}_Y$ and $\underline{A}_X \cap Y$
ii) $(A \cap Y)^{oY}$ and $A^{oX} \cap Y$?