Topology HW3

"Topology generated by"

Gümüşlük Akademisi
July 25th, 2000
Ali Nesin

Let X be a set.

1. For each $i \in I$, let $\mathfrak{I}_i$ be the set of open subsets of some topology on X; thus each $\mathfrak{I}_i$ contains $\varnothing$ and X, is closed under arbitrary unions and finite intersections. Show that $\cap_{i \in I} \mathfrak{I}_i$ is the set of open subsets of a topology, in other words show that

i) $\varnothing \in \cap_{i \in I} \mathfrak{I}_i$ and $X \in \cap_{i \in I} \mathfrak{I}_i$.

ii) $\cap_{i \in I} \mathfrak{I}_i$ is closed under arbitrary unions.

iii) $\cap_{i \in I} \mathfrak{I}_i$ is closed under finite intersections.

2. Let U be a set of subsets of X. Let $\wp$ be the set of open subsets of topologies on X in which the elements of U are open. Show that $\cap\wp$ is the smallest topology on X in which the elements of U are open. This topology is called the **topology generated by** U.

3. Show that open subsets of the topology generated by U are arbitrary unions of the intersections of finitely many elements of U, together with X and $\varnothing$.

4. Show that if U is finite, then the topology generated by U has finitely many open subsets. This is false if we replace "finite" by countable as Question #12 will show.

5. Assume $U = \{\{x\} : x \in X\}$. What are the open subsets of the topology generated by U?

6. Assume $U = \{V \subseteq X : |V| = 2\}$. What are the open subsets of the topology generated by U?

7. Let n be a fixed natural number. Assume $U = \{V \subseteq X : |V| = n\}$. What are the open subsets of the topology generated by U?

8. Let $U = \{V \subseteq X : X \setminus V \text{ is finite}\}$. What are the open subsets of the topology generated by U?

9. Let A be a fixed subset of X and let $U = \{V \subseteq X : A \subseteq V\}$. What are the open subsets of the topology generated by U?

10. Let $A \subseteq X$ and let $U = \{V \subseteq X : A \cap V = \varnothing\}$. What are the open subsets of the topology generated by U?

11. Let $X = \mathbb{R}$ and $U_1 = \{(a, b) : a, b \in \mathbb{R}\}$.The topology generated by U_1 is the usual topology on $\mathbb{R}$. Call this topology $\Im_1$.

12. Let $X = \mathbb{R}$ and $V = \{(a, b) : a, b \in \mathbb{Q}\}$. Show that the topology generated by V is the usual topology $\Im_1$ on $\mathbb{R}$. This shows that a countable set can generate uncountably many open subsets.

13. (Sorgenfrey Line). Let $X = \mathbb{R}$ and $U_2 = \{[a, b) : a, b \in \mathbb{R}\}$. Call $\Im_2$ the topology generated by U_2. Show that every open subset of $\Im_1$ is an open subset of $\Im_2$, but that the converse is false.

14. Let $X = \mathbb{R}$ and $U_3 = \{[a, b] : a, b \in \mathbb{R}\}$. Call $\Im_3$ the topology generated by U_3. Show that every open subset of $\Im_2$ is an open subset of $\Im_3$, but that the converse is false.

Let $\Im$ be a topology on X. If $\Im$ is generated by a set U, then this set U is called a **subbase** of the topology $\Im$. Note that the set of open subsets of $\Im$ is a subbase of $\Im$. A subbase U is called a **base** if U is closed under finite intersections. Note that if U is a base of a topology, then the open subsets of that topology are arbitrary unions of the elements of U.

15. Which of the above subbases is a base?

16. Show that a set U is a base of a topology $\Im$, if for every open subset A of X and every $x \in A$, there is a $B \in U$ such that $x \in B \subseteq A$.

17. Show that the usual topology on $\mathbb{R}^2$ has a countable base.

18. Show that the usual topology on $\mathbb{R}^n$ has a countable base.