Topology HW2

Ali Nesin

July 23rd, 2000

Let (X, d) be a metric space. For $a \in X$ and $r \in \mathbb{R}$, define the open ball with **center** a and **radius** r as $B_d(a, r) = \{x \in X : d(x, a) < r\}$. A subset of X is called **open** (or d-open if confusion may arise) if it is the union of open balls (with varying centers and radii).

1. Let (X, d) be a metric space. Show that the open subsets of X have the following properties:

1a. X and $\emptyset$ are open sets.

1b. Arbitrary union of open sets is open.

1c. Intersection of two open sets is open.

2. Let X be any set and define $d(x, y)$ to be 0 if $x = y$ and to be 1 otherwise. What are the open subsets of X?

3. Let d and δ be two metrics on the same set X. Assume that for any $a \in X$ and any $r \in \mathbb{R}^{>0}$, there is an $s \in \mathbb{R}^{>0}$ such that $B_\delta(a, s) \subseteq B_d(a, r)$. Show that any d-open subset of X is also δ-open.

Two metrics d and δ on the same set X are called **equivalent** if they define the same open sets.

4. Let (X, d) be a metric space. For $x, y \in X$, define $\delta(x, y) = \min(1, d(x, y))$.

4a. Show that δ is a metric on X.

4b. Show that d and δ are equivalent.

5. Let $X = \mathbb{R}^n$ and for $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$**,** define

$$d(x, y) = \sqrt{}(x_1 - y_1)^2 + ... + (x_n - y_n)^2.$$

$$d_1(x, y) = |x_1 - y_1| + ... + |x_2 - y_2|$$

$$d_2(x, y) = \max(|x_1 - y_1|, ..., |x_n - y_n|)$$

5a. Show that these are metrics on $\mathbb{R}^n$.

5b. Show that they are all equivalent.

6a. Show that there are positive constants a and b such that for all $x \in \mathbb{R}^n$,

$$ad_1(x, 0) \le d(x, 0) \le bd_1(x, 0).$$

Here X, d and d_1 are as in question 5.

6b. Find the largest a and smallest b.

7. Let $(x_n)_n$ be a sequence in a metric space. Assume that the subsequences $(x_{2n})_n$, $(x_{2n+1})_n$ and $(x_{3n})_n$ are all convergent. Show that the sequence $(x_n)_n$ is convergent.

8. Prove that the convergence of a real sequence $(x_n)_n$ implies the convergence of the sequence $(|x_n|)_n$. Does the converse hold?