Let $(X, d)$ be a metric space. For $a \in X$ and $r \in \mathbb{R}$, define the open ball with center $a$ and radius $r$ as $B_d(a, r) = \{x \in X : d(x, a) < r\}$. A subset of $X$ is called open (or $d$-open if confusion may arise) if it is the union of open balls (with varying centers and radii).

1. Let $(X, d)$ be a metric space. Show that the open subsets of $X$ have the following properties:
   1a. $X$ and $\emptyset$ are open sets.
   1b. Arbitrary union of open sets is open.
   1c. Intersection of two open sets is open.

2. Let $X$ be any set and define $d(x, y)$ to be 0 if $x = y$ and to be 1 otherwise. What are the open subsets of $X$?

3. Let $d$ and $\delta$ be two metrics on the same set $X$. Assume that for any $a \in X$ and any $r \in \mathbb{R}^+$, there is an $s \in \mathbb{R}^+$ such that $B_{\delta}(a, s) \subseteq B_d(a, r)$. Show that any $d$-open subset of $X$ is also $\delta$-open.

Two metrics $d$ and $\delta$ on the same set $X$ are called equivalent if they define the same open sets.

4. Let $(X, d)$ be a metric space. For $x, y \in X$, define $\delta(x, y) = \min(1, d(x, y))$.
   4a. Show that $\delta$ is a metric on $X$.
   4b. Show that $d$ and $\delta$ are equivalent.

5. Let $X = \mathbb{R}^n$ and for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, define
   
   $d(x, y) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2},$
   
   $d_1(x, y) = \left| x_1 - y_1 \right| + \cdots + \left| x_n - y_n \right|,$
   
   $d_2(x, y) = \max\left( \left| x_1 - y_1 \right|, \ldots, \left| x_n - y_n \right| \right)$

5a. Show that these are metrics on $\mathbb{R}^n$.
5b. Show that they are all equivalent.

6a. Show that there are positive constants $a$ and $b$ such that for all $x \in \mathbb{R}^n$,
   
   $ad_1(x, 0) \leq d(x, 0) \leq bd_1(x, 0)$.

Here $X, d$ and $d_1$ are as in question 5.

6b. Find the largest $a$ and smallest $b$.

7. Let $(x_n)_n$ be a sequence in a metric space. Assume that the subsequences $(x_{2n})_n$, $(x_{2n+1})_n$ and $(x_{3n})_n$ are all convergent. Show that the sequence $(x_n)_n$ is convergent.

8. Prove that the convergence of a real sequence $(x_n)_n$ implies the convergence of the sequence $(\left| x_n \right|)_n$. Does the converse hold?