Topology HW1

Ali Nesin

July 23rd, 2000

1. (Hedgehog space) Let $I = (0, 1]$ and let S be any set. Let $U = I \times S \cup \{(0, 0)\}$. On U^2 define the map d by $d((x, t), (y, s))$ to be $|x - y|$ if $s = t$ and $x + y$ otherwise. Show that d is a metric on U.

2. (Baire Space) Let X be any set. Let Seq(X) be the set of sequences $(x_n)_{n > 0}$ of X. For $x = (x_n)_{n>0}$ and $y = (x_n)_{n>0}$, two elements of Seq(X), define $d(x, y) = 1/k$ where k is the least integer such that $x_i \neq y_i$. Show that d is a metric on Seq(X).

3. (Product Space) Let (X_1, d_1) and (X_2, d_2) be two metric spaces. On $X_1 \times X_2$ define $d((x_1, x_2), (y_1, y_2))$ as

3a. $\sqrt{}d_1(x_1, y_1)^2 + d_1(x_1, y_1)^2$

3b. $\sup(d_1(x_1, y_1), d_2(x_2, y_2))$

3c. $d_1(x_1, y_1) + d_2(x_2, y_2)$

Show that d is a metric on $X \times Y$. Show that the topologies they generate are the same.

4. Let X and Y be two metric spaces. Let $f : X \to Y$ be a **metric preserving map**, i.e. $d(x_1, x_2) = d(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. Show that f is necessarily one-to-one.

5. Find all metric preserving maps from $\mathbb{R}$ into $\mathbb{R}$.

6. For $(a, b) \in \mathbb{R}^2$, define the **translation** $\tau_{(a, b)} : \mathbb{R}^2 \to \mathbb{R}^2$ by $\tau_{(a, b)}(x, y) = (x + a, y + b)$.

For $\theta \in (0, 2\pi]$, define the **rotation** $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by $\rho_\theta(x, y) = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$.

Finally define the **reflection** $i : \mathbb{R}^2 \to \mathbb{R}^2$ by $i(x, y) = (x, -y)$.

Show that $\tau_{(a, b)}$, ρ_θ and i are all distance preserving maps of $\mathbb{R}^2$.

Show that a distance preserving map from $\mathbb{R}^2$ into $\mathbb{R}^2$ is either of the form $\tau_{(a,b)} \circ \rho_\theta$ or of the form $i \circ \tau_{(a,b)} \circ \rho_\theta$ for some $\tau_{(a,b)}$ and ρ_θ.