# Topology I

(Uniform Convergence)
Midterm
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Let $X$ be a set and $(Y, d)$ a metric space. For $f$, $g$ two functions from $X$ into $Y$, define

$$d_\infty(f, g) = \sup\{d(f(x), g(x)) : x \in X\} \in \mathbb{R} \cup \{\infty\}.$$

Consider the set $B(X, Y) = \{f : X \to Y : f(X) \text{ is bounded}\}$.

**1.** Show that $d_\infty(f, g)$ is a real number for $f, g \in B(X, Y)$. (3 pts.)
**2.** Show that $d_\infty$ is a distance on $B(X, Y)$. (3 pts.)

Recall that the sequence $(f_n : X \to Y)_n$ is said to **converge pointwise** to the function $f : X \to Y$ if for all $x \in X$,

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| for all $\varepsilon > 0$ there is an integer $N = N_{\varepsilon, x}$ depending on $\varepsilon$ and $x$ such that $d(f_n(x), f(x)) < \varepsilon$ for all $n > N$. |

The sequence $(f_n)_n$ is said to **converge uniformly** to the function $f$ if the integer $N$ can be found independently of $x$, i.e. if

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| for all $\varepsilon > 0$ there is an integer $N = N_\varepsilon$ depending on $\varepsilon$ alone such that $d(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and $n > N$. |

i.e. if

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| for all $\varepsilon > 0$ there is an integer $N$ such that $d_\infty(f_n, f) < \varepsilon$ for all $n > N$. |

Clearly if $(f_n)_n$ converges uniformly to $f$, then $(f_n)_n$ converges pointwise to $f$. The converse is false as question #5 will show.

**3.** Let $Y = \mathbf{R}$ and let $(a_n)_n$ be a Cauchy sequence of $\mathbf{R}$. Let $f_n$ be the constant function $f_n(x) = a_n$. Show that $f_n$ converges pointwise to a function. Does $f_n$ converge uniformly to a function? (3 pts.)

**4.** Let $(f_n : X \to Y)_n$ be a sequence of functions that converges pointwise to $f : X \to Y$.
**4a.** Show that $(f_n)_n$ converges uniformly to $f$ iff the sequence $(d_\infty(f_n, f))_n$ converges to 0. (3 pts.)
**4b (Cauchy's Criterion).** Show that a sequence $(f_n : X \to \mathbf{R})_n$ of functions converges uniformly to a function iff for all $\varepsilon > 0$ there is an integer $N$ such that for all $n, m > N$, $d_\infty(f_n, f_m) < \varepsilon$. **Hint:** What can $f$ be? (10 pts.)
**4c.** Show that $B(X, \mathbf{R})$ is a complete metric space[1] (with respect to $d_\infty$). (5 pts.)

**5.** Let $X = Y = [0, 1]$. Let $f_n(x) = x^n$.

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[1] This means that every Caucy sequence with respect to the distance $d_\infty$ is convergent.

**5a.** To what function does the sequence $(f_n)_n$ converge pointwise? (2 pts.)

**5b.** Show that the sequence $(f_n)_n$ does not converge uniformly to any function. (5 pts.)

**5c.** Assume now that $X = Y = [0, 1)$. What can you say about the convergence of $(f_n)_n$? (3 pts.)

**5d.** Assume now that $X = Y = [0, a]$ for some $a < 1$. What can you say about the convergence of $(f_n)_n$? (3 pts.)

**5e.** Let $X = [0, 1]$ and $Y = \mathbb{R}$. Let $f_n(x) = n^2x(1 - x)^n$. Show that the sequence $(f_n)_n$ converges pointwise to the zero function, but that the convergence is not uniform. (5 pts.)

**5f.** Now take $X = [0, 1]$ and $f_n(x) = xe^{-nx^2}$. Show that the sequence $(f_n)_n$ converges to the zero function uniformly. (5 pts.)

**6.** Let $(f_n : X \to Y)_n$ be a sequence of functions from a topological space $X$ into a metric space $Y$. Assume the sequence $(f_n)_n$ converges uniformly to $f$.

**6a.** Assume $X = (a, b)$ and that each $f_n$ is continuous at some $c \in X$. Show that $f$ is continuous at $c$ as well. (6 pts.)

**6b.** Assume that each $f_n$ is bounded. Show that $f$ is bounded as well. (3 pts.)

**6c.** Assume that each $f_n$ is bounded. Show that there is a uniform bound for the $f_n$ i.e. for $a \in Y$ there is an $M$ such that $d(f_n(x), a) < M$ for all $n$ and all $x \in X$. (5 pts.)

**6d.** Generalize #6a to general topological spaces. (15 pts.)

**6e.** Show that the set $CB(X, Y)$ of continuous and bounded functions is a closed subset of $B(X, Y)$. (4 pts.)

**7a.** Show that the set of sequences of functions $(f_n : X \to \mathbb{R})_n$ that converge uniformly is a vector space over $\mathbb{R}$. (7 pts.)

**7b.** Assume $(f_n)_n$ and $(g_n)_n$ are sequences from $B(X, \mathbb{R})$ that converge uniformly to $f$ and $g$ respectively. Show that $(f_ng_n)_n$ converges uniformly. **Hint:** See 4a and 6c. (6 pts.)

**7c.** We will now show that the set of sequences of functions $(f_n : X \to \mathbb{R})_n$ that converge uniformly is not closed under product. Take $X = \mathbb{R}$.

Let $f_n(x) = (1+n^{-1})x$ and let

$$g_n(x) = \begin{cases} n^{-1} & \text{if } x = 0 \text{ or } x \notin \mathrm{Q} \\ q + n^{-1} & \text{if } x = p/q \text{ where } p \in \mathrm{Z}, q \in \mathrm{N} \text{ and } (p, q) = 1 \end{cases}$$

Show that $(f_n)_n$ and $(g_n)_n$ converge uniformly on any closed and bounded interval, but that their product does not converge uniformly on any closed interval that contains more than one point. (8 pts.)

**8.** Assume $X$ is also a metric space that each $f_n$ is uniformly continuous[2] and that $(f_n)_n$ converges to $f$. Show that $f$ is uniformly continuous. (5 pts.)

**9 (Dini's Theorem).** Assume $X$ is a compact topological space and $Y = \mathbb{R}$. Assume that each $f_n$ is continuous, that $f_n \geq f_{n+1}$ and that $(f_n)_n$ converges pointwise to a continuous function. Show that the convergence is uniform. (41 pts.)

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[2] Recall that a function $f$ is **continuous** if for all $a \in X$ and for all $\varepsilon > 0$ there is a $\delta_{\varepsilon, a} > 0$ such that if $d(x, a) < \delta$ then $d(f(x), f(a)) < \varepsilon$. The function $f$ is called uniformly continuous if $\delta$ can be chosen independently of $a$, i.e. if for all $\varepsilon > 0$ there is a $\delta_\varepsilon > 0$ such that for all $x$, $a$ if $d(x, a) < \delta$ then $d(f(x), f(a)) < \varepsilon$.