Summer School Topology Midterm

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I.1. Let $(X_i)_i$ be topological spaces. Show that if $\Pi_i X_i$ is Hausdorff (resp. compact) then so is each X_i.

I.2. Show that a closed subset of a compact space is compact.

I.3. Let $f : X \to Y$ be a continuous map between two topological spaces. Show that $f(\overline{A}) \subseteq \overline{f(A)}$. Does the equality hold?

I.4. Show that a continuous image of a compact set is compact. Is the preimage of a compact subset under a continuous map always compact?

II. Let X be a topological space and $A \subseteq X$ a subspace of A.

II.1. Let $K \subseteq A$ be a subset. Show that K is compact in A if and only if K is compact in X.

II.2. Let $K \subseteq X$ be a compact subset of X. Assume that A is closed in X. Show that $K \cap A$ is compact in A.

II.3. Does II.2 still hold if A is not closed?

III. Let $A = \{(x, y) : y < 0\}$ and $B = \{(x, y) : y > 0\}$. Let τ be the set of the following subsets of $\mathbb{R}^2$:

a) Open subsets U of B with the usual Euclidean topology on B,

b) $A \cup U$ where U is as above,

c) $\mathbb{R}^2$.

III.1. Show that τ is a topology on $\mathbb{R}^2$.

III.2. Find the closures of the following subsets in this topology:

III.2.a. $\{(0, 0)\}$.

III.2.b. A.

III.2.c. B.

III.2.d. $\mathbb{R} \times \{0\}$.

III.2.e. $\{0\} \times \mathbb{R}$.

III.2.f. $\{(x, x) : x \in \mathbb{R}\}$.

III.2.g. $\{(x, y) : x^2 + (y - 1)^2 < 1\}$.

III.2.h. $\{(x, y) : x^2 + (y + 1)^2 < 1\}$?

III.2.i. $\{(x, y) : x^2 + (y - 1)^2 = 1\}$.

III.2.k. $\{(x, y) : x^2 + (y + 1)^2 = 1\}$?

III.3. Describe all the compact subsets of $\mathbb{R}^2$ with respect to τ.

IV. Let X be a Hausdorff topological space.

IV.1. Let K be a compact subset of X. Show that K is closed. Give a counterexample to this statement when X is not Hausdorff.

IV.2. Let K be a compact subset of X and $a \in X \setminus K$. Show that there are disjoint open subsets U and V such that $K \subseteq U$ and $a \in V$.

IV.3. Let K_1 and K_2 be two disjoint compact subsets. Show that there are two disjoint open subsets U_1 and U_2 of X such that $K_i \subseteq U_i$.

IV.4. Assume now X is a metric space with respect to the metric d. Let K_1 and K_2 be as above. Show that $\inf\{d(x, y) : x \in K_1 \text{ and } y \in K_2\} > 0$.

V. 1. Show that a compact subset of a Hausdorff space is closed.

V. 2. Show that a compact subset of a metric space is bounded.

V. 3. Let $A = \{(x, y) : y < 0\}$ and $B = \{(x, y) : y > -1\}$. Let σ be the topology generated by A and the open subsets U of B (with the usual Euclidean topology on B). Show that A is compact but $\underline{A}$ is not compact in this topology.

VI. [Alexandroff One-Point Compactification]. Let X be a locally compact topological space. That means that for any $x \in X$ there is compact neighborhood of x, i.e. for any $x \in X$, there is a compact subset K containing an open subset U such that $x \in U \subseteq K$. Let ∞ be an element not in X. On $Y = X \cup \{\infty\}$ consider the topology generated by the open subsets of X and the complements in Y of compact subsets of X.

VI.1. Show that Y is a compact space.

VI.2. Show that the initial topology of X is the topology on X restricted from that of Y.

Y is called the **one-point compactification** of X.

VI.3. Find well-known topological spaces which are homeomorphic to the one-point compactifications of $\mathbb{R}$ and $\mathbb{R}^2$.