Math 111 Final (Topology)
2006 Fall
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As you can see from the points attached to each question, the questions are not difficult at all; they are in fact rather easy. The only difficulty lies in grasping the concepts. Once you understand what the concepts are saying, the rest will be easy. Please take your time and read the definitions carefully, find your own examples.

Do not try to do all the questions in order to amass points because I may take away points for absurd answers.

Write carefully, clearly and legibly. Make full sentences with at least one subject and a verb. Do not use symbols such as $\Rightarrow$, $\forall$.

Let X be a set. Let $\tau \subseteq \wp(X)$ be such that

T1. $\varnothing \in \tau$ and $X \in \tau$.

T2. If U, $V \in \tau$ then $U \cap V \in \tau$. (I.e. τ is closed under finite intersections).

T3. If $\sigma \subseteq \tau$, then $\cup\, \sigma \in \tau$. (I.e. τ is closed under arbitrary unions).

Such a τ is called a ***topology*** on X. The elements of τ are called ***open*** subsets of X (for the topology τ). The pair (X, τ) is called a ***topological space.***

Part I.

1. *Let τ be the set of cofinite subsets of X together with $\varnothing$. Show that τ is a topology on X. This is called the* ***cofinite topology*** *on X.* (4 pts.)

Answer: By definition $\varnothing \in \tau$. Since $X^c = \varnothing$ and $\varnothing$ is finite, X is cofinite, so $X \in \tau$. Thus T1 is satisfied.

Now we check T2. Let U, $V \in \tau$. If U or V is $\varnothing$, then $U \cap V = \varnothing \in \tau$. Assume now that neither U nor V is $\varnothing$. Then U and V are cofinite, i.e. U^c and V^c are finite. Hence their union $U^c \cup V^c$ is finite. Since $(U \cap V)^c = U^c \cup V^c$, this shows that $U \cap V$ is cofinite, i.e. $U \cap V \in \tau$.

Finally we check T3. Let $\sigma \subseteq \tau$. If $\cup\sigma = \varnothing$ then $\cup\sigma \in \tau$. Assume $\cup\sigma \neq \varnothing$. Then σ must contain a nonempty subset A of τ. Then A is cofinite and $A \subseteq \cup\sigma \subseteq X$. Thus $\cup\sigma$ is also cofinite (even more cofinite than A!)

2. *Show that the intersection of any nonempty set of topologies is a topology on X.* (3 pts.)

Proof: Let Σ be a set of topologies on X. Let $\tau = \cap\Sigma = \cap_{\sigma\in\Sigma}\, \sigma$. Since each element σ of Σ is a topology, they all contain $\varnothing$ and X. Thus $\varnothing$ and X are elements of $\cap_{\sigma\in\Sigma}\, \sigma = \tau$

Now we check T2. Let U, $V \in \tau$. Thus U, $V \in \sigma$ for all $\sigma \in \Sigma$. Since each $\sigma \in \Sigma$ is a topology, this implies that $U \cap V \in \sigma$. Hence $U \cap V \in \cap_{\sigma\in\Sigma}\, \sigma = \tau$.

Finally T3. Let $\alpha \subseteq \tau$. Then $\alpha \subseteq \sigma$ for all $\sigma \in \Sigma$. Since each element σ of Σ is a topology, $\cup\alpha \in \sigma$. Thus $\cup\alpha \in \cap_{\sigma\in\Sigma}\, \sigma = \tau$.

3. *Find the smallest and the largest topologies* (*for the inclusion*) *for X. The smallest topology on X is called the* ***coarsest topology*** *on X and the largest topology is called the* ***discrete topology*** *on X.* (4 pts.)

Answer: Each topology must contain $\varnothing$ and X. On the other hand it is easy to check that $\tau_0 = \{\varnothing, X\}$ is a topology. Hence τ_0 is the smallest topology on X.

Certainly $\tau = \wp(X)$ is a topology on X. Thus it is the largest topology on X.

4. *Given any $\sigma \subseteq \wp(X)$ show that there is a smallest topology $\langle\sigma\rangle$ that contains σ (as a subset).* (10 pts.)

Proof: By #2, the intersection of toplogies is a topology. And by #3, $\wp(X)$ is a topology on X that contains σ. Thus the set of topologies on X that contains σ is a nonempty set. Take the intersection of all topologies on X that contain σ. By #2, this is a topology. It certainly contains σ as a subset. Thus it can only be the smallest topology that contains σ. We set $\langle\sigma\rangle = \cap\{\tau \subseteq \wp(X) : \sigma \subseteq \tau \text{ and } \tau \text{ is a topology}\}$.

5. *Show that if $\sigma \subseteq \wp(X)$ is closed under finite intersections then*

$$\langle\sigma\rangle = \{\cup\alpha : \alpha \subseteq \sigma\} \cup \{X\}.$$

(10 pts.)

Proof: Let $\varphi = \{\cup\alpha : \alpha \subseteq \sigma\} \cup \{X\}$.

Since $\sigma \subseteq \langle\sigma\rangle$, it is clear that for any $\alpha \subseteq \sigma$, α is also a subset of $\langle\sigma\rangle$. Since $\langle\sigma\rangle$ is a topology, this implies by T3 that $\cup\alpha \in \langle\sigma\rangle$. This shows that $\{\cup\alpha : \alpha \subseteq \sigma\} \subseteq \langle\sigma\rangle$ and so $\varphi \subseteq \langle\sigma\rangle$ (because, $\langle\varphi\rangle$, being a topology, contains X).

On the other hand, for any $A \in \sigma$, if we take $\alpha = \{A\}$, we see that $A = \cup\alpha \in \{\cup\alpha : \alpha \subseteq \sigma\}$. Hence $\sigma \subseteq \varphi$.

Thus $\sigma \subseteq \varphi \subseteq \langle\sigma\rangle$.

Accordingly, if we can show that φ is a topology then we would necessarily have $\varphi = \langle\sigma\rangle$ because $\langle\sigma\rangle$ is by definition the smallest topology that contains σ. So we should check T1, T2 and T3 hold for φ.

Taking $\alpha = \emptyset$, we see that $\emptyset = \cup\alpha \in \{\cup\alpha : \alpha \subseteq \sigma\} \subseteq \varphi$. By definition $X \in \varphi$. This shows that T1 holds for φ.

Let us check T2. Let $A, B \in \varphi$. If one of A or B is X, then $A \cap B$ is either A or B, so that $A \cap B \in \varphi$. If neither A or B is X, then $A = \cup\alpha$ and $B = \cup\beta$ for some $\alpha, \beta \subseteq \sigma$. Thus $A \cap B = (\cup\alpha) \cap (\cup\beta) = \cup\gamma$ where $\gamma = \{a \cap b : a \in \alpha, b \in \beta\}$. But since α and β are subsets of σ and σ is closed under finite intersections, for each $a \in \alpha$ and $b \in \beta$, we have $a \cap b \in \sigma$. Thus $\gamma \subseteq \sigma$, proving that $A \cap B \subseteq (\cup\alpha) \cap (\cup\beta) = \cup\gamma \in \{\cup\alpha : \alpha \subseteq \sigma\} \subseteq \varphi$.

Finally we check T3. Let $\delta \subseteq \varphi$. If $X \in \delta$, then $\cup\delta = X \in \varphi$. Thus we may assume that $X \notin \delta$. Therefore $\delta \subseteq \{\cup\alpha : \alpha \subseteq \sigma\}$. Then $\delta = \cup\beta$ where $\beta = \cup\{\alpha \subseteq \sigma : \cup\alpha \in \delta\}$. Since β is a subset of σ, this shows that $\delta = \cup\beta \in \{\cup\alpha : \alpha \subseteq \sigma\} \subseteq \varphi$.

6. *Given an arbitrary $\sigma \subseteq \wp(X)$, can you describe the elements of $\langle\sigma\rangle$ by using the elements of σ (as above)?* (10 pts.)

Answer: Let $\sigma_1 = \{A_1 \cap \ldots \cap A_n : n \in \mathbb{N} \text{ and } A_i \in \sigma\}$. It is clear that $\sigma \subseteq \sigma_1$, so that $\langle\sigma\rangle \subseteq \langle\sigma_1\rangle$. It is also clear that σ_1 is closed under finite intersections, so that, by #6,

$$\langle\sigma_1\rangle = \{\cup\alpha : \alpha \subseteq \sigma_1\} \cup \{X\}.$$

Thus $\langle\sigma\rangle \subseteq \langle\sigma_1\rangle = \{\cup\alpha : \alpha \subseteq \sigma_1\} \cup \{X\}$.

We claim that $\langle\sigma_1\rangle \subseteq \langle\sigma\rangle$. For this, it is of course enough to show that $\sigma_1 \subseteq \langle\sigma\rangle$. Let us take an arbitrary element $A_1 \cap \ldots \cap A_n$ of σ_1. Here each A_i is assumed to be in σ. Thus each A_i is in $\langle\sigma\rangle$. Since $\langle\sigma\rangle$ is a topology, $\langle\sigma\rangle$ is closed under finite intersections, implying that $A_1 \cap \ldots \cap A_n \in \langle\sigma\rangle$.

Part II. *Let (X, τ) and (Y, σ) be two topological spaces. A map $f : X \to Y$ is called* ***continuous*** *if for any open subset V of Y, $f^{-1}(V)$ is an open subset of X.*

7. *Let X be a set. Find a topology on X such that for any topological space* (Y, σ) *and any map $f : X \to Y$, the map f is continuous.* (6 pts.)

Answer: On X take the discrete topology. Thus any subset of X (for this topology) is open. Therefore for any function $f : X \to Y$ and for any open subset V of Y, $f^{-1}(V)$ is an open subset of X.

8. *Let Y be a set. Find a topology on Y such that for any topological space* (X, τ) *and any map $f : X \to Y$, the map f is continuous.* (6 pts.)

Answer: On Y take the coarsest topology. Thus Y has only two open subsets: $\varnothing$ and Y itself. Thus for V we have only two choices: If $V = \varnothing$, then $f^{-1}(V) = \varnothing$ is open in X (whatever the topology of X is) and if $V = Y$, then $f^{-1}(V) = X$ is open in X.

9. *Show that a constant map between topological spaces is always continuous.* (4 pts.)

Proof: Let X and Y be two topological spaces and $f : X \to Y$ be a constant map. Let $b \in Y$ be such that $f(x) = b$ for all $x \in X$. Let V be an open subset of Y. If $b \notin V$, then $f^{-1}(V) = \varnothing$ and is open in X. If $b \in V$, then $f^{-1}(V) = X$ and is open in X.

10. *Show that the identity map is a continuous function of a fixed topological space.* (2 pts.)

Proof: Let (X, τ) be a topological space. For V, an open subset of X, $\mathrm{Id}^{-1}(V) = V$. Thus $\mathrm{Id}^{-1}(V)$ is open for any open subset V of X. This means that Id is continuous. (Note that this would be false if the domain X and the range X have different topologies).

11. *Show that the composition of continuous maps is continuous.* (4 pts.)

Proof: Let X, Y and Z be three topological spaces and $f : X \to Y$ and $g : Y \to Z$ be two continuous maps. Let $W \subseteq Z$ be an open subset. Then $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$. Since g is continuous, $g^{-1}(W)$ is an open subset of Y. Since f is continuous, $f^{-1}(g^{-1}(W))$ is an open subset of X. Hence $(g \circ f)^{-1}(W)$ is open in X, proving that $g \circ f$ is continuous.

12. *Let* (Y, σ) *be a topological space, X a set and $f : X \to Y$ be a map. Show that there is a smallest topology on X that makes f continuous.* (10 pts.)

Proof: Let τ be the topology on Y. Let σ be the topology that we want to find on X. (We still don't know that σ exists). Clearly, for f to be continuous, for any $V \in \tau$, we must have $f^{-1}(V) \in \sigma$. Thus for f to be continuous, σ must contain $f^{-1}(V)$ for all $V \in \tau$, i.e. we must have $\{f^{-1}(V) : V \in \tau\} \subseteq \sigma$.

But it is easy to check that $\{f^{-1}(V) : V \in \tau\}$ is itself a topology. Thus $\sigma = \{f^{-1}(V) : V \in \tau\}$ is the smallest topology that maks f continuous.

13. *Let* (Y, σ) *be a topological space, X a subset of Y and $i : X \to Y$ the inclusion map. Find the smallest topology on X that makes i continuous. This topology on X is called the* ***restricted topology*** *on X.* (6 pts.)

Proof: By above, this topology must be $\{i^{-1}(V) : V \in \sigma\}$. But $i^{-1}(V) = V \cap X$. Thus the open subsets of X must all be of the form $V \cap X$ for some open subset V of Y.

14. *Let* (X, τ) *and* (Y, σ) *be two topological spaces. Let π_1 and π_2 be the two projection maps from $X \times Y$ onto X and Y respectively. Thus $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Find the smallest topology on $X \times Y$ that makes π_1 and π_2 continuous.* (10 pts.) *This topology on*

$X \times Y$ is called the ***product topology****. Show that the projections of any open subset of $X \times Y$ for the product topology are open.* (5 pts.)

Proof: The smallest topology on $X \times Y$ must contain $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ for any open subsets U and V of X and Y respectively. Thus the smallest topology must also contain $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = (U \times Y) \cap (X \times V) = U \times V$ for any open subsets U and V of X and Y respectively. Such a set $U \times V$ is called an ***open rectangle***. Unfortunately, the set $\{U \times V : U \in \tau, V \in \sigma\}$ of open rectangles is not a topology, because although it is closed under finite intersetions, this set is not closed under arbitrary unions. As in #5, we must take arbitrary unions of open rectangles. Thus the open subsets of the smallest topology that makes both projection maps are arbitrary unions of open rectangles.
The second part is easy.

15. *Let $(X_i, \tau_i)_i$ be a family of topological spaces. Find the least topology on $\Pi_i X_i$ that makes all the projections π_i continuous.* (12 pts.)

Proof: As above... Let I be the index set. For $i(1), ..., i(n) \in I$ and for open subsets $U_{i(1)} \subseteq X_{i(1)}, ..., U_{i(n)} \subseteq X_{i(n)}$, let

$$V(i(1), ..., i(n), U_{i(1)}, ..., U_{i(n)}) = \{x \in \Pi_i X_i : x_{i(1)} \in U_{i(1)}, ..., x_{i(n)} \in U_{i(n)}\}.$$

The sets of this form are closed under finite intersections, but are not closed under unions. Close them under taking the unions. Call an open subset of $\Pi_i X_i$ open if it is an arbitrary union of sets of the form $V(i(1), ..., i(n), U_{i(1)}, ..., U_{i(n)})$ as above. As in #5, this is a topology and it is certainly the lest topology that makes all the projection maps continuous.

Part III. *In this part a set X and a topology τ on X are fixed. We will say that X is a* ***topological space****.*

16. *Let A be a subset of X. Show that there is a largest open subset of A. We let $A°$ denote this largest subset of A.* (5 pts.)

Proof: There is at least one open subset in A, namely $\emptyset$. Now take the union of all the open subsets of A. Being a union of open sets, this is an open set. Since it is a subset of A, it must be the largest open subset of A. Thus $A° = \cup\{U : U$ open and $U \subseteq A\}$. Note that any open subset of A is a subset of $A°$.

17. *Assume τ is the cofinite topology, show that either $A° = A$ or $A = \emptyset$.* (4 pts.)

Proof: If U is cofinite, then U is open and so $U = U°$. Otherwise, U cannot have a cofinite subset, thus the only open subset of U is $\emptyset$ and so $A° = \emptyset$.

18. *Show that $A = A°$ if and only if A is open.* (2 pts.)

Proof: Since $A°$ is open, if $A = A°$, A is clearly open. If A is open, A is of course the largest open subset of itself, so that $A = A°$.

19. *Show that $A°° = A°$.* (2 pts.)

Proof: Since $A°$ is open, $A°° = A°$ by above.

20. *Show that if $A \subseteq B$ then $A° \subseteq B°$.* (4 pts.)

Proof: We have $A° \subseteq A \subseteq B$. Thus $A° \subseteq B$. Since $A°$ is open subset of B and since $B°$ is the largest open subset of B, we must have $A° \subseteq B°$.

21. *Show that* $A° \cap B° = (A \cap B)°$. (4 pts.)
Proof: Since $A° \cap B°$ is open and since $A° \cap B° \subseteq A \cap B$, we must have $A° \cap B° \subseteq (A \cap B)°$. Now the reverse inclusion: Since $(A \cap B)° \subseteq A \cap B \subseteq A$, $(A \cap B)°$ is an open subset of A. Therefore $(A \cap B)° \subseteq A°$. Similarly $(A \cap B)° \subseteq B°$. Thus $(A \cap B)° \subseteq A° \cap B°$.

22. *Show that* $A° \cup B° \subseteq (A \cup B)°$. (2 pts.) *Show that the equality may not always hold.* (5 pts.)
Proof: Since $A \subseteq A \cup B$, by #20, $A° \subseteq (A \cup B)°$. Similarly $B° \subseteq (A \cup B)°$. Thus $A° \cup B° \subseteq (A \cup B)°$. To show that the inequality may not hold, let us take X to be an infinite set, e.g. $X = \mathbb{N}$, and the cofinite topology on it. Take A be an infinite subset of X which is not cofinite, e.g. $A = 2\mathbb{N}$. Let $B = X \setminus A$. Then $A° = B° = \varnothing$ but $(A \cup B)° = X° = X$.

23. *Let* Y *be another topological space and* $f : X \to Y$ *be a continuous map. Let* $B \subseteq Y$. *Find the set theoretic relationship between* $f^{-1}(B°)$ and $f^{-1}(B)°$. (6 pts.)
Proof: Since $B°$ is open and f continuous, $f^{-1}(B°)$ is open as well. Since $f^{-1}(B°) \subseteq f^{-1}(B)$, this implies that $f^{-1}(B°) \subseteq f^{-1}(B)°$. To see that the equality may not hold, take $X = Y = \mathbb{N}$ with the cofinite topology and f to be the constant 0-map. By #9 f is continuous. Take $B = \{0\}$. Then $B° = \varnothing, f^{-1}(B) = \mathbb{N}$.

Part IV. *In this part a topological space X is fixed. A subset F of X is called* **closed** *if its complement is open.*

24. *Show that* $\varnothing$ *and* X *are closed.* (1 pts.)
Proof: Since $\varnothing^c = X \setminus \varnothing = X$ and X is open $\varnothing$ is closed. Similarly X is closed.

25. *Show that the union of two closed subsets of* X *is closed.* (2 pts.)
Proof: Let C and D be two closed sets. Then C^c and D^c are open. Therefore their intersection $C^c \cap D^c$ is open as well. Since $(C \cup D)^c = C^c \cap D^c$, this implies that $C \cup D$ is closed.

26. *Show that the intersection of any set of closed subsets of* X *is closed.* (3 pts.)
Proof: Let $\Im$ be a set of closed subsets of X. Then $(\cap\mathcal{F})^c = (\cap_{C \in \mathcal{F}} C)^c = \cup_{C\in\mathcal{F}} C^c$. But C^c is open for each $C \in \mathcal{F}$. Thus $\cup_{C\in\mathcal{F}} C^c$ is open. Therefore $\cap\mathcal{F}$ is closed.

27. *Let* A *be a subset of* X. *Show that there is a smallest closed superset of* A. *We let* $\underline{A}$ *denote this smallest closed subset that contains* A. (4 pts.)
Proof: X is a closed superset of A. Let $\underline{A}$ be the intersection of all closed supersets of A. Clearly $A \subseteq \underline{A}$. By above $\underline{A}$ is closed. Clearly $\underline{A}$ is a subset of any closed superset of A. Thus $\underline{A}$ is the smallest closed superset of A.

28. *Show that* $A = \underline{A}$ *if and only if* A *is closed.* (2 pts.)
Proof: Since $\underline{A}$ is closed, if $A = \underline{A}$, A is closed. Conversely, if A is closed, then A is of course the smallest closed superset of itself. Thus $A = \underline{A}$.

29. *Show that* $\underline{\underline{A}} = \underline{A}$. (2 pts.)
Proof: Since $\underline{A}$ is closed, by the above result, $\underline{\underline{A}} = \underline{A}$.

30. *Show that if $A \subseteq B$ then $\underline{A} \subseteq \underline{B}$.* (3 pts.)
Proof: Since $A \subseteq B \subseteq \underline{B}$, we have $A \subseteq \underline{B}$. Thus $\underline{B}$ is a closed superset of A. Thus $\underline{A} \subseteq \underline{B}$.

31. *Show that $\underline{A} \cup \underline{B} = \underline{A \cup B}$.* (4 pts.)
Proof: $\underline{A} \cup \underline{B}$ is a closed superset of $A \cup B$. Thus $\underline{A \cup B} \subseteq \underline{A} \cup \underline{B}$. On the other hand, by #30, $\underline{A} \subseteq \underline{A \cup B}$. Similarly $\underline{B} \subseteq \underline{A \cup B}$. Thus $\underline{A} \cup \underline{B} \subseteq \underline{A \cup B}$.

32. *Show that $\underline{A \cap B} \subseteq \underline{A} \cap \underline{B}$* . (2 pts.) *Show that the equality may not always hold.* (4 pts.)
Proof: By #30, $\underline{A \cap B} \subseteq \underline{A}$. Similarly $\underline{A \cap B} \subseteq \underline{B}$. Thus $\underline{A \cap B} \subseteq \underline{A} \cap \underline{B}$. In the cofinite topology, the closed subsets are only the finite subsets and the space X itself. Take $X = \mathbb{N}$, $A = 2\mathbb{N}$, $B = 2\mathbb{N} + 1$. Then $\underline{A} = \underline{B} = \mathbb{N}$ and $A \cap B = \varnothing$.

33. *Let Y be another topological space. Let $f : X \to Y$ be a map. Show that f is continuous if and only if the inverse image of any closed subset of Y is closed in X.* (5 pts.)
Proof: Assume f is continuous. Let $F \subseteq Y$ be a closed subset. We have $f^{-1}(F)^c = f^{-1}(F^c)$ and is open in X because F^c is open in Y and f is continuous.
Conversely, assume that the inverse image of any closed subset of Y is closed in X. Let V be open in Y. Then $f^{-1}(V) = f^{-1}(V^c)^c$. Since V^c is closed, $f^{-1}(V^c)$ is closed as well. Thus its complement $f^{-1}(V^c)^c$ is open. Therefore $f^{-1}(V)$, which is equal to $f^{-1}(V^c)^c$ is open as well.

34. *Let Y be another topological space. Let $f : X \to Y$ be a continuous map. Let $B \subseteq Y$. Find the set theoretic relationship between $f^{-1}(\underline{B})$ and $\underline{f^{-1}(B)}$.* (4 pts.)
Proof: By above $f^{-1}(\underline{B})$ is closed. Certainly $f^{-1}(B) \subseteq f^{-1}(\underline{B})$. Thus $\underline{f^{-1}(B)} \subseteq f^{-1}(\underline{B})$. The equality may not hold: Take $X = Y = \mathbb{N}$ with the cofinite topology. Take f to be the constant 0-map. f is continuous by #9. Let $B = \mathbb{N} \setminus \{1\}$. Then $\underline{B} = \mathbb{N}$, $f^{-1}(\underline{B}) = \mathbb{N}$ but $f^{-1}(B) = \varnothing$.

Part IV. *A topological space X is called* ***connected*** *if it is not the union of two disjoint nonempty open subsets. A subset A of a topological space X is called* ***connected*** *if A is connected for the restricted topology.*

35. *Let $A \subseteq X$ be a connected subspace of X. Show that $\underline{A}$ is connected.* (15 pts.)
Proof: Assume $\underline{A}$ is not connected. Thus there are two disjoint and nonempty open subsets C and D (in the restricted topology) of $\underline{A}$ such that $\underline{A} = C \cup D$. Then, by the definition of restriction topology, $C = \underline{A} \cap U$ and $D = \underline{A} \cap V$ for some open subsets U and V of X. Now $A = A \cap \underline{A} = A \cap (C \cup D) = (A \cap C) \cup (A \cap D) = (A \cap \underline{A} \cap U) \cup (A \cap \underline{A} \cap V) = (A \cap U) \cup (A \cap V)$. Since A is connected and $A \cap U$ and $A \cap V$ are disjoint open subsets of A, one of them must be empty, without loss of generality say $A \cap U = \varnothing$. Then $A \subseteq U^c$ and U^c is closed. Hence $\underline{A} \subseteq U^c$ and $C = \underline{A} \cap U = \varnothing$, a contradiction.

36. *Show that the relation "$x \equiv y$ iff x and y belong to a connected subset of X" is an equivalence relation on X.* (15 pts.)
Proof: Let $x \in X$. Since the set $\{x\}$ is singleton, it must be connected. Thus $x \equiv x$.
Assume $x \equiv y$. Thus x and y belong to the same connected subset A. Then y and x also belong to the connected subset A. Hence $y \equiv x$.

Assume now that $x \equiv y$ and $y \equiv z$. Let A and B be two connected sets such that $x, y \in A$ and $y, z \in B$. Thus $x, z \in A \cup B$. It remains to show that $A \cup B$ is connected. Note that $y \in A \cap B$, so that $A \cap B \neq \emptyset$. We will show the following:

Lemma. *If A and B are connected and $A \cap B \neq \emptyset$, then $A \cup B$ is connected.*
Proof: Let C and D be two disjoint and nonempty open subsets (in the restricted topology) of $A \cup B$ such that $A \cup B = C \cup D$. Thus $A = (A \cap C) \cap (A \cap D)$. Note that $A \cap C$ and $A \cap D$ are disjoint open subsets (in the restricted topology) of A. Hence either $A \cap C$ or $A \cap D$ is empty. Without loss of generality we may assume that $A \cap C = \emptyset$. Thus $A \subseteq D$. Similarly, $B = (B \cap C) \cap (B \cap D)$ and either $B \cap C$ or $B \cap D$ is empty. If $B \cap C = \emptyset$ then $B \subseteq D$ and so $A \cup B \subseteq D$ and $C = \emptyset$, a contradiction. Thus $B \cap D = \emptyset$; but then $B \subseteq C$ and so $A \cap B \subseteq C \cap D = \emptyset$, a contradiction again. This proves the lemma.

37. *Show that each equivalence class for the above relation is a maximal connected subset of X* (20 pts.) *and is closed* (5 pts.). *Each equivalence class is called a* ***connected component*** *of X. Show that each connected component is also open if there are finitely many of them.* (6 pts.) *Give a counterexample when there are infinitely many connected components.* (10 pts.)
Proof: We need a lemma similar to the one above:

Lemma. *Let $(A_i)_{i \in I}$ be a family of connected subsets of X. Assume that for $i \neq j$ in I, $A_i \cap A_j \neq \emptyset$. Then $\cup_{i \in I} A_i$ is connected.*
Proof: Let C and D be two disjoint and nonempty open subsets (in the restricted topology) of $\cup_{i \in I} A_i$ such that $\cup_{i \in I} A_i = C \cup D$. Thus $A_i = (A_i \cap C) \cap (A_i \cap D)$. Note that $A_i \cap C$ and $A_i \cap D$ are disjoint open subsets (in the restricted topology) of A_i. Hence either $A_i \cap C$ or $A_i \cap D$ is empty. Without loss of generality we may assume that $A_i \cap C = \emptyset$. Thus $A_i \subseteq D$. Now let $j \neq i$. Similarly, we have $A_j = (A_j \cap C) \cap (A_j \cap D)$ and either $A_j \cap C$ or $A_j \cap D$ is empty. If $A_j \cap D = \emptyset$; but then $A_j \subseteq C$ and so $A_i \cap A_j \subseteq C \cap D = \emptyset$, a contradiction. Thus $A_j \cap C = \emptyset$ and $A_j \subseteq D$ all $j \in I$. This shows that $\cup_{i \in I} A_i \subseteq D$ and so $C = \emptyset$, a contradiction. This proves the lemma.

Continued. For each $y \in [x]$, choose a connected set A_x that contains x and y. By the lemma above the set $\cup_{y \in [x]} A_x$ is connected and it certainly contains $[x]$. If C is connected and contains $\cup_{y \in [x]} A_x$ then for any element $c \in C$, c and x would be in the same connected set C, so $c \equiv x$ and $c \in [x]$. This shows that $\cup_{y \in [x]} A_x = C = [x]$. Thus $[x]$ is a maximal connected subset of X.
By #35, $\underline{[x]}$ is connected. Since $[x]$ is maximal, this shows that $[x] = \underline{[x]}$ and so $[x]$ is closed.

It is **false** that each $[x]$ is open: Take $X = \mathbb{R} \times \mathbb{R}$. Let the open subsets of X be the complements of finitely many vertical lines of X. This really efines a topology. In this topology, for each $x \in X$, $[x]$ is the vertical line passing through x and is not open.
Note that in the above counterexample, there are infinitely many equivalence classes (i.e. connected components). But the question is true if X has finitely many connected components:
Clearly, $X = \cup_{x \in X} [x]$ and $[x]^c = \cup\{[y] : y \in X \text{ and } [y] \neq [x]\}$ and this set is closed if the union is taken over a fnite set.

38. *Show that the product of two connected spaces is connected for the product topology.* (20 pts.)

Proof: Let X and Y be two connected topological spaces. Assume $X \times Y = U \cup V$ for two open nonempty disjoint subsets of $X \times Y$. I am too tired...