Topology Final Exam
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1. *Let X be a metric space. Show that for any $x \in X$ there is a countable set of open subsets $V_n(x)$ ($n \in \mathbb{N}$) containing x such that for any open subset U containing x there is an $n \in \mathbb{N}$ such that $V_n \subseteq U$.*

Proof: For a rational number $q \in \mathbb{Q}^{>0}$, let $V_q(x) = B(x, q)$.

2. *Let I be an uncountable (index) set. Let X be a topological space whose topology is not the coarsest topology. Show that the product (Tychonoff) topology on $\Pi_I X$ is not metrisable.*

Proof: We will show that for some $x \in \Pi_I X$ there is no countable open subsets $(V_n(x))_n$ as in the first question. Let $a \in X$. Define $x \in \Pi_I X$ to be the element whose coordinate x_i is a. Let $V \neq X$ be an open subset of X containing a. For each $i \in I$ define $U_i = \{y \in \Pi_I X : y_i \in V\}$. Then each U_i is an open subset containing x. For any countable family $(V_n(x))_n$ of open subsets containing x, the set $\{i \in I : \text{pr}_i(V_n(x)) \neq X\}$ being finite for all $n \in \mathbb{N}$, the set $\{i \in I : \text{pr}_i(V_n(x)) \neq X \text{ some } n \in \mathbb{N}\}$ will be countable. Thus its complement $J = \{i \in I : \text{pr}_i(V_n(x)) = X \text{ all } n \in \mathbb{N}\}$ will be uncountable. If $V_n(x) \subseteq U_i$, then, $\text{pr}_i(V_n(x)) \subseteq \text{pr}_i(U_i) = V \neq X$ and so $i \notin J$.. Thus not all the U_i's can contain one of the $V_n(x)$'s.

3. *Let X be a set. Let F be the set of real (or complex) valued functions from X. For $f \in$* Func$(X, \mathbb{R})$ *define $\|f\| = \sup\{\min\{|f(x)|, 1\} : x \in X\}$ and $d(f, g) = \|f - g\|$. Show that (F, d) is a metric space.*

Convergence for this metric is called ***uniform convergence****.*

Proof: Note first that sup exists because the numbers are bounded by 1.

All the properties of the metric are clear except may be for the triangular inequality. Let f, g and h be three functions. We have to show that $d(f, g) \leq d(f, h) + d(h, g)$, i.e. that $\|f - g\| \leq \|f - h\| + \|h - g\|$. Replacing $f - h$ by f and $h - g$ by g, we are brought to the inequality $\|f + g\| \leq \|f\| + \|g\|$. If one of $\|f\|$ or $\|g\|$ is 1, the inequality is clear. Assume $\|f\| < 1$ and $\|g\| < 1$. Then $|f(x)| < 1$ and $|g(x)| < 1$ for all $x \in X$ and $\|f\| = \sup\{|f(x)| : x \in X\}$ and $\|g\| = \sup\{|g(x)| : x \in X\}$. Hence, $\|f + g\| = \sup\{\min\{|f(x)+g(x)|, 1\} : x \in X\} \leq \sup\{\min\{|f(x)|+|g(x)|, 1\} : x \in X\} \leq \sup\{|f(x)|+|g(x)| : x \in X\} \leq \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\} = \|f\| + \|g\|$.

4. *Let $(f_n)_n$ be a sequence of functions from a set X into $\mathbb{R}$ (or $\mathbb{C}$). Suppose that there is a sequence $(M_n)_n$ of real numbers such that* **a)** *$\Sigma_n M_n$ converges and* b) *$|f_n(x)| \leq M_n$ for all $x \in X$ and $n \in \mathbb{N}$. Show that $\Sigma_n f_n$ converges uniformly.*

Proof: It is enough to show that the series $(\Sigma_{i \leq n} f_i)_n$ is Cauchy in the sup metric $\| \ \|$. Let $\varepsilon > 0$. We have to find a natural number N such that for $n \geq m > N$, $\|\Sigma_{i \leq n} f_i - \Sigma_{i \leq m} f_i\| < \varepsilon$.

Note first that $\|f_n\| \leq M_n$.

Since $\Sigma_n M_n$ converges, the sequence $(\Sigma_{i \leq n} M_i)_n$ is Cauchy. Hence there is a natural number N such that for all $n \geq m > N$, $\|\Sigma_{m < i \leq n} M_i\| = \|\Sigma_{i \leq n} M_i - \Sigma_{i \leq m} M_i\| < \varepsilon$. Thus for all $n \geq m > N$, $\|\Sigma_{i \leq n} f_i - \Sigma_{i \leq m} f_i\| = \|\Sigma_{m < i \leq n} f_i\| \leq \Sigma_{m < i \leq n} \|f_i\| \leq \|\Sigma_{m < i \leq n} M_i\| < \varepsilon$.

5. *Show that the limit of a uniformly convergent sequence of real (or complex) valued continuous functions from a metric space is continuous.*

Proof: Let $(f_n)_n$ be a uniformly sequence of continuous functions from a metric space (X, d) into $\mathbb{R}$ (or $\mathbb{C}$). Let f be the uniform (hence also the pointwise) limit of the sequence. Let $a \in A$. Let $\varepsilon > 0$. We have to find $\delta > 0$ such that if $d(a, x) < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Since $\lim_{n\to\infty} f_n = f$, there is an N such that for all $n > N$, $\|f_n - f\| < \varepsilon/3$. It follows that for all $n > N$ and $x \in X$, $|f_n(x) - f(x)| < \varepsilon/3$.

Let $n = N + 1$. Since f_n is continuous at a there is a $\delta > 0$ such that $|f_n(x) - f_n(a)| < \varepsilon/3$ whenever $d(x, a) < \delta$.

Now, for any $x \in X$ for which $d(x, a) < \delta$, $|f(x) - f(a)| \le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

6. *Show that a continuous function from a compact space into a metric space is uniformly continuous.*

Proof: Recall that a function $f : X \to Y$ (between metric spaces X and Y) is uniformly continuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $d(f(x), f(y)) < \varepsilon$ for all $x, y \in X$ for which $d(x, y) < \delta$. Suppose X is compact and $f : X \to Y$ is continuous. Let $\varepsilon > 0$. Since f is continuous, for each $x \in X$ there is a $\delta(x) > 0$ such that $d(f(x), f(y)) < \varepsilon/2$ for all $y \in B(x, \delta(x))$. Since $X = \cup_{x\in X} B(x, \delta(x)/2)$ and X is compact, there are $x_1, ..., x_n \in X$ such that $X = B(x_1, \delta(x_1)/2) \cup ... \cup B(x_n, \delta(x_n)/2)$. Let $\delta = \min\{\delta(x_1), ..., \delta(x_n)\}/2$. Let $x, y \in X$ be such that $d(x, y) < \delta$. Let $i = 1,..., n$ be such that $x \in B(x_i, \delta(x_i)/2)$. Then $d(y, x_i) \le d(y, x) + d(x, x_i) < \delta + \delta(x_i)/2 \le \delta(x_i)/2 + \delta(x_i)/2 = \delta(x_i)$ and so $d(f(y), f(x_i)) < \varepsilon/2$. Now $d(f(x), f(y)) \le d(f(x), f(x_i)) + d(f(x_i), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$.

Given a sequence $(a_n)_n$ *recall that* $\limsup a_n$ *is defined as* $\lim_{n\to\infty} \sup\{a_m : m > n\}$.

7. *Let* $\Sigma_n a_n x^n$ *be a power series.* Let $R = 1/\limsup |a_n|^{1/n}$. *Show that* $\Sigma_n a_n x^n$ *converges absolutely for* $|x| < R$ *and diverges for* $|x| > R$. (*Recall that if* $b_n \ge 0$ *for all n then* $\limsup b_n$ *is defined as the limit of the nonincreasing sequence* $\sup\{b_i : i > n\} \in \mathbb{R}^{\ge 0} \cup \{\infty\}$ *and* $1/\infty$ *and* $1/0$ *are defined to be* 0 *and* ∞ *respectively. R is called the radius of convergence of the series* $\Sigma_n a_n x^n$).

Proof: Let $r = \limsup |a_n|^{1/n} < 1$. Let $\varepsilon = (1 - r)/2$ and $\rho = r + \varepsilon$. Then $\rho = r + \varepsilon = r + (1-r)/2 = (1 + r)/2 < 1$. Since $\varepsilon > 0$, the nonincreasing sequence $\sup\{|a_n|^{1/n} : i > n\} < r + \varepsilon$ after a while. i.e. $|a_n|^{1/n} \in [r, r + \varepsilon)$

8. Let $0 \le S < R$. Show that $\Sigma_n a_n x^n$ converges uniformly on $\{x : |x| \le S\}$.

9. Show that the radius of convergence of the power series $\Sigma_{n\ge 0} a_n x^n$ and $\Sigma_{n>0} n a_n x^{n-1}$ are the same.

10. Let $\Sigma_n a_n x^n$ be a power series and R its radius of convergence. Show that the series is differentiable at any x such that $|x| < R$.