Topology Final (Math 222)
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1. *Let X be a topological space. A collection $\Im$ of subsets of X is said to be* ***locally finite*** *in X if every point of X has a neighborhood that intersects only finitely many elements of $\Im$. Let $\Im$ be locally finite.*

1a. *Show that any subcollection of $\Im$ is locally finite.*

1b. *Show that* $\{\mathrm{cl}(A) : A \in \Im\}$ *is locally finite.*

1c. *Show that* $\mathrm{cl}(\cup_{A \in \Im} A) = \cup_{A \in \Im} \mathrm{cl}(A)$.

Proof. 1a. Clear.

1b. Let $x \in X$. Let U be an open neighborhood of x that intersects only finitely many $A \in \Im$. Suppose $U \cap \mathrm{cl}(A) \neq \emptyset$ for some $A \in \Im$. Let $y \in U \cap \mathrm{cl}(A)$. Since U is an open neighborhood of y, $U \cap A \neq \emptyset$. Thus there are finitely many such A.

1c. Let $A \in \Im$. Since $A \subseteq \cup_{A \in \Im} A \subseteq \mathrm{cl}(\cup_{A \in \Im} A)$, $\mathrm{cl}(A) \subseteq \mathrm{cl}(\cup_{A \in \Im} A)$. Hence $\cup_{A \in \Im} \mathrm{cl}(A) \subseteq \mathrm{cl}(\cup_{A \in \Im} A)$. Conversely suppose there is an $x \in \mathrm{cl}(\cup_{A \in \Im} A) \setminus \cup_{A \in \Im} \mathrm{cl}(A)$. Let U be any open subset containing x which intersects only finitely many $A \in \Im$. Let $\wp = \{A \in \Im : A \cap U \neq \emptyset\}$. Then $x \in \mathrm{cl}(\cup_{A \in \Im} A) = \mathrm{cl}((\cup_{A \in \Im \setminus \wp} A) \cup (\cup_{A \in \Im} A)) = \mathrm{cl}(\cup_{A \in \Im \setminus \wp} A) \cup (\cup_{A \in \wp} \mathrm{cl}(A))$. Since $x \notin \mathrm{cl}(A)$ for any $A \in \Im$, $x \in \mathrm{cl}(\cup_{A \in \Im \setminus \wp} A)$. Thus $U \cap (\cup_{A \in \Im \setminus \wp} A) \neq \emptyset$, a contradiction.

2. *Let $\Im$ be a collection of subsets of the topological space X. A collection $\wp$ of subsets of X is said to be a* ***refinement*** *of $\Im$ if for each $B \in \wp$ there is an $A \in \Im$ such that $B \subseteq A$.*

Let $\Im$ be the following collection of subsets of $\mathbb{R}$: $\Im = \{(n, n + 2) : n \in \mathbb{Z}\}$. *Which of the following refine $\Im$:*

$\wp_1 = \{(x, x + 1) : x \in \mathbb{R}\}$,

$\wp_2 = \{(n, n + 3/2) : n \in \mathbb{Z}\}$,

$\wp_1 = \{(x, x + 3/2) : x \in \mathbb{R}\}$.

Answer: For $\wp_1$: Let $x \in \mathbb{R}$. Then $[x] \leq x < x + 1 < ([x] + 1) + 1 = [x] + 2$, so that $(x, x + 1) \subseteq ([x], [x] + 2)$. Hence $\wp_1$ refines $\Im$.

For $\wp_2$: $(n, n + 3/2) \subseteq (n, n + 2)$. Hence $\wp_2$ refines $\Im$ as well.

For $\wp_3$: Take $x = 3/4$. Then $(x, x + 3/2) = (3/4, 9/4) \in \wp_3$ and none of the intervals of $\Im$ contains it. Thus $\wp_3$ does not refine $\Im$.

3. A *space is said to be* ***normal*** *if for every pair of disjoint closed subsets A and B there are open disjoint subsets U, V containing A and B respectively. Show that every metric space is normal.*

Proof: Let $a \in A$. Consider the subset $\{d(a, b) : b \in A\}$ of $\mathbb{R}^{\geq 0}$.

If 0 were in the closure of this set, then we could find a sequence $(b_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} d(a, b_n) = 0$. Then we can choose a convergent subsequence $(c_n)_n$ of $(b_n)_n$. We have necessarily $\lim_{n\to\infty} c_n = a$. But then $a \in B$, a contradiction.

Thus there is an $\alpha(a) > 0$ such that $d(a, b) > \alpha(a)$ for all $b \in B$.

Similarly, for all $b \in B$, there is a $\beta(b) > 0$ such that $d(a, b) > \beta(b)$ for all $a \in A$.

Now consider $U = \cup_{a\in A} B(a, \alpha(a)/2)$ and $V = \cup_{b\in B} B(b, \beta(b)/2)$. Assume $x \in B(a, \alpha(a)/2) \cap B(b, \beta(b)/2)$ for some $a \in A$ and $b \in B$. Assume also, without loss of generality, that $\alpha(a) \leq \beta(b)$. Then $d(a, b) \leq d(a, x) + d(x, b) < \alpha(a)/2 + \beta(b)/2 \leq \beta(b)$, a contradiction.

4. *Show that a closed subspace of a normal space is normal.*

Proof: Let Y be a closed subspace of X. Let A and B be two disjoint closed subsets of Y. Then A and B are closed in X. Hence there are disjoint open subsets U and V of X containing A and B respectively. Then $U \cap Y$ and $V \cap Y$ are disjoint open subsets of Y containing A and B respectively.

5. *Let X be a totally ordered set. Let $\Im$ be the collection of all sets of the following types*:

(1) *Open intervals* $(a, b) = \{x \in X : a < x < b\}$.

(2) *Intervals of the form* $[a_0, b)$ *where* a_0 *is the smallest element of* X (*if there is any*).

(3) *Intervals of the form* $(b, a_1]$ *where* a_1 *is the largest element of* X (*if there is any*).

Consider the topology generated by $\Im$. Show that this topology is Hausdorff.

Proof: Let $a, b \in X$ be distinct. Without loss of generality assume $a < b$.

Assume first there is a $c \in (a, b)$.

If a is the least element and b is the largest element then $[a, c)$ and $(c, b]$ separate a and b.

If a is the least element and b is not the largest element then let $d > b$. Then $[a, c)$ and (c, d) separate a and b.

If neither a nor b are the extremal points then let $d < a$ and $b < e$. Then (d, c) and (c, e) separate a and b.

Now assume that $(a, b) = \emptyset$.

If a is the least element and b is the largest element then $[a, b) = \{a\}$ and $(a, b] = \{b\}$ separate a and b.

If a is the least element and b is not the largest element then let $d > b$. Then $[a, b) = \{a\}$and (a, d) separate a and b.

If neither a nor b are the extremal points then let $d < a$ and $b < e$. Then (d, b) and (a, e) separate a and b.

6. *Show that a topological space is Hausdorff if and only if the diagonal*

$$\Delta = \{(x, x) : x \in X\}$$

is closed.

Proof: Suppose X is Hausdorff. Let $x \neq y$. Let U and V be two open disjoint subsets of X containing x and y respectively. Then $U \times V$ is an open subset of $X \times X$ not intersecting Δ. Hence Δ is closed.

Conversely suppose Δ is closed. Let $x \neq y$. Then $(x, y) \notin \Delta$ and (x, y) is in the open subset Δ^c. So there is a basic open subset $U \times V$ containing (x, y) not intersecting Δ. Hence U and V separate x and y.

7. *Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at precisely one point.*

Solution: Let $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. We claim that f is continuous at 0 only.

Continuity at 0: Let $\varepsilon > 0$ be any. Let $\delta = \varepsilon$. Then for all $|x - 0| < \delta$, $|f(x) - f(0)| = |f(x)| = 0$ or $x \in (-\varepsilon, \varepsilon)$.

Discontinuity at $a \in \mathbb{Q}^{>0}$. Let $\varepsilon = a$. Let $\delta > 0$ be any. Then for $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $|x - a| < \delta$, $|f(x) - f(a)| = a \notin (-\varepsilon, \varepsilon)$.

Discontinuity at $a \in \mathbb{R}^{>0} \setminus \mathbb{Q}$. Let $\varepsilon = a/2$. Let $\delta > 0$ be any. Let $\alpha = \inf\{\varepsilon, \delta\}$. Then for $x \in \mathbb{Q}$ such that $|x - a| < \alpha$, $|f(x) - f(a)| = x \notin (-\varepsilon, \varepsilon)$.

8. *Let X be a topological space. Let C be a connected subset of X.*

8a. *Show that* cl(C) *is connected.*

8b. *Let $A \subseteq X$ be connected. Show that there is a maximal connected subset B containing X.*

8c. *Show that every maximal connected subset is closed.*

8d. *Show that X is a disjoint union of maximal connected subsets.*

Proof: 8a. Suppose $\mathrm{cl}(C) \subseteq U \cup V$ for two open and disjoint subsets U and V. Then $C \subseteq U \cup V$. Hence either $C \subseteq U$ or $C \subseteq V$. In the first case $\mathrm{cl}(C) \subseteq V^c$ and in the second case $\mathrm{cl}(C) \subseteq U^c$. Therefore either $\mathrm{cl}(C) \cap V = \varnothing$ or $\mathrm{cl}(C) \cap U = \varnothing$.

8b. Let B be the union of connected subsets of X containing A. We claim that B is connected; this will prove the assertion. Let $B \subseteq U \cup V$ where U and V are open and disjoint. Let C be any connected subspace containing A. Then $C \subseteq U \cup V$ and thereforr either $C \subseteq U$ or $C \subseteq V$. Since the same holds for A, all such C must be either in U or in V. Hence either $B \subseteq U$ or $B \subseteq V$.

8c. Let C be a maximal connected subset. Then by 8a, cl(C) is connected as well. Hence $C = \mathrm{cl}(C)$.

8d. Let C and D be two connected subsets. Assume $C \cap D \neq \varnothing$. Let $x \in C \cap D$. Then $\{x\}$ is connected and C and D are maximal connected subsets containing $\{x\}$. Hence $C = D$ by 8b. If we let C_x denote the maximal connected subset containing x then X is the disjoint union of sets of the form C_x for some $x \in X$. (Set $x \equiv y$ iff $C_x = C_y$. Then this is an equivalence relation and the equivalence classe of x is C_x).

9. *Let X have countable basis. Let A be an uncountable subset of X. Show that uncountably many points of A are limit points of A.*