Topology Exam

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1. *Let (X, τ) be a topological space. Let $Z \subseteq Y \subseteq X$. Let τ_Y and τ_Z be the topologies induced by X on Y and Z respectively. Let σ be the topology induced by τ_Y on Z. Show that $\tau_Z = \sigma$.*

Proof: Let $C \subseteq Z$ be σ-open. Then there is an τ_Y-open subset B of Y such that $C = B \cap Z$. Since $B \in \tau_Y$, there is an open subset $A \in \tau$ such that $B = A \cap Y$. Hence $C = B \cap Z = (A \cap Y) \cap Z = A \cap Z$ and so $C \in \tau_Z$.

Assume now $C \subseteq Z$ is τ_Z-open. Then there is a τ-open subset A of X such that $C = A \cap Z$. Then $C = A \cap Z = (A \cap Z) \cap Y = (A \cap Y) \cap Z$. Since $A \cap Y \in \tau_Y$, C is in σ.

2. *Let A and B be two subsets of a topological space. Let $Y \subseteq X$. Show that the set $Y \cap (A \cup B)$ is open (resp. closed) in $A \cup B$ if and only* if $Y \cap A$ is open (resp. closed) in A and $Y \cap B$ is open (resp. closed) in B.

Proof: Suppose $Y \cap (A \cup B)$ is open (resp. closed) in $A \cup B$. Then $Y \cap A$ is open (resp. closed) in A and $Y \cap B$ is open (resp. closed) in B by Question 1.

Conversely suppose $Y \cap A$ is open (resp. closed) in A and $Y \cap B$ is open (resp. closed) in B. Then $Y \cap A = Y \cap A \cap U$ and $Y \cap B = Y \cap B \cap V$ for some open subsets U and V of X. Hence, $Y \cap (A \cup B) = (Y \cap A) \cup (Y \cup B) = (Y \cap A \cap U) \cup (Y \cap B \cap V) = Y \cup ((A \cap U) \cup (B \cap V))$

3. *Let $(A_i)_i$ be a locally finite[1] cover of a topological space X. Let $B \subseteq X$. Show that B is open (resp. closed) in X if and only if each of the subsets $B \cap A_i$ is open (resp. closed) in A_i.*

[1] i.e. for each $x \in X$ there is an open subset U containing x such that $U \cap A_i \neq \emptyset$ only for finitely many i.