Topology Resit Exam (Math 112)
September, 2004,
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Answers without proof or justification will not be accepted.

Do not use symbols such as ∀, ∃, ⇒. Make full sentences, in either language you feel confortable with.

1. *Let $f : X \to Y$ be a function from a topological space X into a topological space Y. Let A be the set of points of X where f is discontinuous. Let g be the restriction of f to $X \setminus A$. Is g continuous everywhere? Prove or disprove.*

Proof: You should first notice that $X \setminus A$ has the induced subset topology, otherwise the question has no meaning. Let $x \in X \setminus A$. By definition f is continuous at x. Let V be an open subset of Y containing $g(x)$. Since $g(x) = f(x)$ and f is continuous at x, there is an open subset U of X such that $x \in U \subseteq f^{-1}(V)$. Then $x \in U \cap (X \setminus A) \subseteq f^{-1}(V) \cap (X \setminus A) = g^{-1}(V)$. (Here, the last equality is to be proven). Since $U \cap (X \setminus A)$ is an open subset of $X \setminus A$, it follows that g is continuous at x.

2. *Let X be a topological space. A point x of X is called* ***isolated*** *if $\{x\}$ is open. Let A be the set of isolated points of X. Does the space $X \setminus A$ has any isolated points? Prove or disprove.*

Answer: Yes, the space $X \setminus A$ may have isolated points. Here, once again $X \setminus A$ is endowed with the induced subset topology. Let $X = \{1/n : n = 1, 2, 3,...\} \cup \{0\}$, with the topology induced from the Euclidean topology of $\mathbb{R}$. Then each $1/n$ is an isolated point of X but 0 is not an isolated point. Thus $X \setminus A = \{0\}$ and 0 is an isolated point of $X \setminus A$.

3. *Let X be a topological space and $f : X \to X$ be a function. Suppose that $f \circ f$ is continuous. Is f necessarily continuous? Prove or disprove.*

Answer: Wrong! Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ otherwise. f is not continuous. But $f \circ f$ is the constant 0 function and so is continuous.

4. *Let X and Y be topological spaces. Let $f : X \to Y$ be a function. Let $g : X \to f(X)$ be defined as f. Thus, $f(x) = g(x)$ for all $x \in X$. Does the continuity of one of f or g imply the continuity of the other? Prove or disprove both.*

Answer: Continuity of one imply the continuity of the other. Suppose first f is continuous. Let W be an open subset of $f(X)$. Thus $W = V \cap f(X)$ for an open subset V of Y. Then $g^{-1}(W) = g^{-1}(V \cap f(X)) = g^{-1}(V) \cap g^{-1}(f(X)) = f^{-1}(V) \cap X = f^{-1}(V)$ and is open in X. Suppose now g is continuous. Let V be an open subset of Y. Then $f^{-1}(V) = f^{-1}(V \cap f(X)) = g^{-1}(V \cap f(X))$ and so $f^{-1}(V)$ is an open subset of X because $V \cap f(X)$ is an open subset of $f(X)$.

5. *A map $f : X \to Y$ from a topological space X into a topological space Y is called* ***closed*** *(resp.* ***open****) if $f(C)$ is closed (resp. open) in Y for any closed (resp. open) subset C of X.*

5a. *Let $A \subseteq X$ be a subset endowed with the induced topology. When is the inclusion map $i : A \to X$ is open (resp. closed)?*

Answer: If i is open (resp. closed), then $i(A)$ is open (resp. closed) in X, i.e. A is open (resp. closed) in X. We show that this condition is enough to show that i is open (resp. closed). Let S be an open (resp. closed) subset of A. Then $S = T \cap A$ for some open (resp. closed) subset T of X. Then $i(S) = S = T \cap A$ is open (resp. closed) in A.

5b. *Let* $f : X \to Y$ *be a map from a set* X *into a topological space* Y. *Endow* X *with the least topology that makes* f *continuous. Is* f *necessarily open? Is* f *necessarily closed? Prove or disprove.*

Answer: Recall that open subsets of X are the inverse images of open subsets of Y under f. It follows quite easily that closed subsets of X are the inverse images of closed subsets of Y under f. Let S be an open (resp. closed) subset of X. Then $S = f^{-1}(T)$ for some open (resp. closed) subset of Y. Thus $f(S) = f(f^{-1}(T)) \subseteq T$, and the equality holds if f is onto. But if f is not onto the equality may not hold and in this case it seems that f may not be open (resp. closed) in Y.

Indeed let f be a constant function, say b. On Y take the topology where the open subsets are Y and the ones that do not contain the element b. Then the topology on X is the least topology where only $\varnothing$ and X are open. Then $f(X) = \{b\}$ and is not open. So in this case f is not an open map.

Now let $c \in Y \setminus \{b\}$ and consider as open subsets of Y the set Y itself and the subsets that do not contain c. Thus closed subsets of Y are $\varnothing$ and the subsets that contain c. Then $f(X) = \{b\}$ and is not closed. So in this case f is not a closed map.

5c. *Let* $f : X \to Y$ *be a continuous function. Let* $g : X \to f(X)$ *be defined by* $g(x) = f(x)$. *Supposing that* f *is open (resp. closed), does this imply that* g *is open (resp. closed)? Supposing that* g *is open (resp. closed), does this imply that* f *is open (resp. closed)?*

Proof: Let Y be such that for some $b \in Y$, $\{b\}$ is not open in Y. E.g. $Y = \mathbb{R}$ with the usual topology. Let f be the constant function whose value is b. Then f is continuous but not open because $f(X)$ is not open in Y. On the other hand $g : X \to \{b\}$ is open, because $\{b\}$ is open in $\{b\}$. We may also choose Y so that $\{b\}$ is not closed in Y (e.g. $Y = \mathbb{R}$, where open subsets are only $\varnothing$ and $\mathbb{R}$.) Then the constant b function f is continuous but not closed because $f(X)$ is not closed. But g is closed.

But if f is open (resp. closed), then g is open (resp. closed) because if $C \subseteq X$ is open (resp. closed), then $g(C) = f(C)$.

5d. *Let* X *and* Y *be two topological spaces. Endow* $X \times Y$ *with the product topology. Let* $\pi : X \times Y \to X$ *be the first projection. Is* π *necessarily open? Is* π *necessarily closed? Prove or disprove.*

Answer: π is not necessarily closed. For example, take $X = Y = \mathbb{R}$ with the usual topology and $C = \{(x, y) : xy = 1\}$. Then C is closed, but $\pi(C) = \mathbb{R} \setminus \{0\}$ is not closed. On the other hand π is open, because every open subset of $X \times Y$ is a union of sets of the form $U \times V$ where U and V are open in X and Y respectively, so that the image of an open subset of $X \times Y$ under π is the union of open subsets U that appear in its definition and is open.

5e. *Find a continuous map* $f : X \to X$ *from a topological space* X *into itself which is not closed.*

5f. *Find a continuous map* $f : X \to X$ *from a topological space* X *into itself which is not open.*

5g. *Find a continuous map* $f : X \to X$ *from a topological space* X *into itself which is neither open nor closed.*

Answer: Take $X = \mathbb{R}$ with its usual topology and $f(x) = 1/(1+x^2)$. Then $f(\mathbb{R}) = (0, 1]$ and is neither closed nor open.

5h. *Find a closed function* $f : X \to X$ *from a topological space* X *into itself which is not continuous.*

Answer: Take $X = \mathbb{R}$ and define $f(x) = 0$ if $x \le 0$ and $f(x) = 1$ if $x > 0$. Then f is closed but not continuous.

5i. *Find an open function* $f : X \to X$ *from a topological space* X *into itself which is not continuous.*

Answer: Let $X = \mathbb{R}^{\geq 0}$ with the usual topology. For $x \in \mathbb{R}$ let $[x]$ denote the largest integer such that $[x] \leq x$. Let $f(x) = x - [x]$ regarded as a function from X into X. Then f is open but not continuous.

5j. *Let* X, Y, Z *be three topological spaces and* $f : X \to Y$ *and* $g : Y \to Z$ *be two functions. Show that if* $g \circ f$ *is open (resp. closed) and* g *is continuous and surjective then* g *is open (resp. closed). Find the limitations of this result.*

Proof: Let V be an open subset of Y. Let $U = f^{-1}(V)$. Then U is open because f is continuous and $g(f(U))$ is open because $g \circ f$ is open. But since f is surjective $f(U) = V$. Hence $g(V) = g(f(U))$ is open. The rest is left to you.

5k. *Let* X, Y, Z *be three topological spaces and* $f : X \to Y$ *and* $g : Y \to Z$ *be two functions. Show that if* $g \circ f$ *is open (resp. closed) and if* g *is continuous and injective then* f *is open (resp. closed). Find the limitations of this result.*

Proof: Let $U \subseteq X$ be an open subset. Then $g(f(U))$ is open because $g \circ f$ is open and $g^{-1}(g(f(U)))$ is open because g is continuous. But $g^{-1}(g(V)) = V$ for all $V \subseteq Y$ because g is injective. Thus $f(U)$ is open.

6. *Let* X *and* Y *be topological spaces. Let* $f : X \to Y$ *be a function. Show that* f *is continuous and closed (see # 5) if and only if* $f(\underline{A}) = \underline{f(A)}$ *for every subset* A *of* X. *(Here* $\underline{A}$ *denotes the closure of* A*).*

Proof: Suppose f is continuous and closed. Let A be a subset of X. Then, since f is closed, $f(\underline{A})$ is closed, and since it contains $f(A)$, $\underline{f(A)} \subseteq f(\underline{A})$. Since f is continuous, $f^{-1}(\underline{f(A)})$ is closed, and since it contains A, it also contains $\underline{A}$. Hence $f(\underline{A}) \subseteq \underline{f(A)}$.

Conversely suppose that $f(\underline{A}) = \underline{f(A)}$ for every subset A of X. To show that f is continuous, it is enough to assume only that $f(\underline{A}) \subseteq \underline{f(A)}$ for all $A \subseteq X$. Indeed, let F be a closed subset of Y. Then $f(\underline{f^{-1}(F)}) \subseteq \underline{f(f^{-1}(F))} \subseteq \underline{F} = F$, so $\underline{f^{-1}(F)} \subseteq f^{-1}(F)$. This shows that $\underline{f^{-1}(F)} = f^{-1}(F)$ and $f^{-1}(F)$ is closed. This shows that f is continuous. Let now $A \subseteq X$ be closed. Then $f(A) = f(\underline{A}) = \underline{f(A)}$ and so $f(A)$ is closed.